ELEMENTARY ALGEBRA (041)

This review covers Elementary Algebra concepts found on the Rutgers math placement exam. The content in this PDF was created by the Math department at Rutgers-Camden.
Contents
1-D Linear Inequalities .................................................................................................................. 2
Ratio is a Comparison ..................................................................................................................... 5
Commutative and Associative ........................................................................................................ 7
Division by Zero ............................................................................................................................. 8
Evolution of Numbers ................................................................................................................... 10
Expressions Equations Literal ....................................................................................................... 12
Factoring ........................................................................................................................................ 15
Gaussian Elimination ................................................................................................................... 17
Graphing Lines ............................................................................................................................. 23
Integer Reactions ........................................................................................................................ 26
Like Terms Exponents .................................................................................................................. 28
Order of Operations Polynomials ............................................................................................... 31
Polynomial Division ..................................................................................................................... 34
Rationales vs Irrationals .............................................................................................................. 39
Second Level Cycle Factoring ..................................................................................................... 43
Simple One Cycle Factoring ........................................................................................................ 46
Simplifying Roots ........................................................................................................................ 48
The Concept of LCD ..................................................................................................................... 52
The Concept of Percent ............................................................................................................... 56
Word Problems ............................................................................................................................ 58


**1-D Linear Inequalities**

In 1-D (dimension) $X=3$ locates a single point (moment in time) on the numberline. Singular inequalities are bounded on one side and unbounded on the other. $X \geq 3$ & $X \leq 3$ own the boundary of 3 so the solidified disc is used to designate this. Whereas $X>3$ & $X<3$ do NOT own the boundary so the open(empty) disc symbolizes this. Interval notation uses a [ when the boundary is included and ( when it is not included.

A dual inequality is bounded on both ends so $0 \leq X < 4$ says $X$ lies between 0 (inclusively) and 4 (non inclusively). These actually satisfy two inequalities simultaneously. $X < 4$ and $X \geq 0$ which agree in between 0 and 4. Interval notation is [0,4)

Whereas $x \leq 0$ or $x > 4$ cannot be combined into a dual inequality. Absolute value is a distance measuring machine. Think of it as the odometer in your car which does not know the direction you went only how far from zero you travelled. Absolute value inequalities using < are just compact forms of dual inequalities. So $|x-5| < 3$ says:

$-3 < x-5 < 3$ which means $x-5$ lies between -3 & 3. If they have > as the connection then this says $|x|$ is above or below a designated value. So $|x-5| > 3$ says $x-5 > 3$(above 3) or $x-5 < -3$ (below -3).

So if $k=3$

$|3x-6| < 9$ says $3x-6$ lies between -9 and 9. -9 < $3x-6$ <9 which says: -3 < $3x$ <15 so -1 < $x$ < 5 .

$|3x-6| > 9$ says $3x-6$ lies above 9 or below -9 so $3x-6 > 9$ sees $x > 5$ or $3x-6 < -9$ so $x < -1$
For 2-D inequalities first graph the line noticing whether the boundary is included. You own the fence when \( \leq \) is used, but you do not own the fence when \( < \) is used. When you put up a fence it is inspected by the town in which you live to prevent encroaching on someone else’s property and recorded on your deed as a solid line \( \leq \) indicating that you own the fence. On the neighbor’s deed it is recorded as a dotted line \( < \) indicating they do not own the fence but the fence exists between the properties.

Once the fence is located then it’s time to determine which side of the fence you own. Use \((0,0)\) to test where to shade. If \((0,0)\) agrees with the inequality signal then shade where it is since you own that property including the blade of grass on \((0,0)\). If it disagrees with the signal then shade other side from where \((0,0)\) lies since you own the property away from where \((0,0)\) is located.

\[
Y \leq 3X - 1
\]

For this one, graph the line \(y=3X-1\) using \((0,-1)\) as the y-axis hit and a slope of 3 to 1. It is solidified because the boundary is included (you own the fence). Then since \((0,0)\) sees \(0 \leq 3(0) -1\) says \(0 \leq -1\) which is false (disagrees with the signal) you shade the other side relative to where \((0,0)\) lies.

\[
\frac{3}{2}X - 3
\]

For this one, graph of the line generated by \(y = \frac{3}{2}X - 3\) using \((0,-3)\) as the y-axis hit and the slope of \(\frac{3}{2}\) which is 3 to 2. It is dotted because you do not own the boundary (fence). Then testing \((0,0)\) sees \(0 > \frac{3}{2}(0) -3\) says \(0 > -3\) which is true so you
shade where (0,0) lies because you own the blade of grass on (0,0) and all the grass up to the fence. (but not the fence)

For this one, graph $3X + 4Y = 3$ by locating the fence (which you do not own therefore dotted) by homing in on the x-axis and y-axis hits. At $x=0$: $3X+4Y = 3$ sees $4Y=3$ so $Y = \frac{3}{4}$ for $(0, \frac{3}{4})$ as the x-axis hit. For $Y = 0$: $3X + 4Y = 3$ sees $3X=3$ so $X=1$ for $(1,0)$ as the y-axis hit. Then testing $(0,0)$ sees $3(0) + 4(0) < 3$ says $0 < 3$ which is true so you shade where $(0,0)$ lies for once again your property is on the side where $(0,0)$ lies. You can’t use $(0,0)$ if it lies on the fence so move off and use any other point not on the fence.

When more than one condition is to be met simultaneously then graph each condition on the same axis to see where they converge and agree. This is how a bounded area is generated.
**Ratio is a Comparison**

A **ratio** is a comparison between two entities that have the same measure. For example: 2 of your marbles in comparison to 3 of my marbles.

| A **rate** is a comparison of two things of different measure.  
| For example 1 foot is swappable for 12 inches, 1yd is exchangeable for 3ft.  
| These are **global** rates which says they are not dependent upon time nor place.  
| | 1 ft | 1 yd |
| | 12 in | 3 ft |
| A **local** rate is dependent upon time or place. For example I can buy 3 lbs of apples for $1.50 today in my grocery store, but this does not mean that someone in Colorado gets this price or that this price will be available next week to me.  
| | 3 lbs | 1.50 cost |
| You might swap me, 1 baseball for 5 marbles, but this does not mean this applies to everyone in every location.  
| | 1 baseball | 5 marbles |

A **proportion** is an equality involving two rates of comparison. The cross product determines if and when two rates are equal. To understand why a proportion is balanced consider the following.

If I have 5:7ths as \( \frac{5}{7} \) then I can multiply it by any form of ONE without disturbing its place in space. You can think of multiplying by ONE as painting a room a different color but does not change the dynamics: volume, length, width, height etc of the room nor its position in the building. It simply owns the same place in space. It simply changes the perspective of the room.

ONE comes in infinite colors. We may need \( \frac{2}{3}, \frac{3}{5}, \frac{4}{5} \) etc. to do the job at hand. So if you want \( \frac{5}{7} \) to be seen in 21sts then

\[
\frac{5}{7} \times \frac{3}{3} = \frac{15}{21} \quad \text{So} \quad \frac{5}{7} = \frac{15}{21}
\]

The cross products are balanced because internally we simply multiplied by ONE. Analogously if a component of a proportion is missing then we assume we are looking for the number that causes the balance within therefore assume the cross products are balanced. So consider: \( \frac{5}{7} = \frac{15}{w} \)

Since you assume the number you seek causes the balance, you have the cross products which see: \( 5w = 7(15) \) and proceed to solve by seeing \( \frac{(7)(15)}{5} \). You can internally cut down on the calculation by getting rid of the \( 5/5 \) within this calculation reducing it to \( 7(3) \) for 21. This reducing insight is very powerful in large numerical contexts.

For example if you have: \( \frac{7}{48} = \frac{54}{72} \)

This leads to \( \frac{(48)(54)}{72} \) which after reducing \( \frac{9}{9} \) and \( \frac{8}{8} \) you see \( (6)(6) \) for 36 rather than multiplying 48 times 54 then dividing by 72.
Regardless of which corner is missing, the pattern of the cross product process is that you always multiply the two DIAGONALLY ACROSS from each other divided by the third component.

Given one rate of comparison two questions can be formed. The ALIGNMENT of the third piece of information is critical for it determines which question is being addressed.

If I know I can cover 15 sq ft of patio with 90 brick then the two questions that can be formed are:

A. How many brick will I need to cover 180 sq ft of patio?
   \[
   \frac{15 \text{ sq ft}}{90 \text{ brick}} = \frac{180 \text{ sq ft}}{? \text{ brick}}
   \]
   which leads to 1080 brick.

B. How many sq ft will 180 brick cover?
   \[
   \frac{15 \text{ sq ft}}{90 \text{ brick}} = \frac{? \text{ sq ft}}{180 \text{ brick}}
   \]
   which leads to 30 brick.

If the third piece of information is NOT ALIGNED PROPERLY then the other question actually gets addressed.

**Now for general eyes.** If I know I can get ‘b’ basketballs for $10 then represent:

A) How many basketballs I can get for ‘d’ dollars.
   \[
   \frac{b \text{ basketballs}}{10 \text{ dollars}} = \frac{? \text{ basketballs}}{d \text{ dollars}}
   \]
   processed reveals \( \frac{bd}{10} = ? \)

B) How much will ‘d’ basketballs cost.
   \[
   \frac{b \text{ basketballs}}{10 \text{ dollars}} = \frac{d \text{ basketballs}}{? \text{ dollars}}
   \]
   processed reveals \( \frac{10d}{b} = ? \)

If I can get ‘q’ quibbles for ‘t’ tribbles:

A) Represent how many tribbles I can get for ‘s’ quibbles.
   \[
   \frac{q \text{ quibbles}}{t \text{ tribbles}} = \frac{s \text{ quibbles}}{? \text{ tribbles}}
   \]
   leads to \( \frac{ts}{q} = ? \)

B) Represent how many quibbles I can get for ‘s’ tribbles.
   \[
   \frac{q \text{ quibbles}}{t \text{ tribbles}} = \frac{? \text{ quibbles}}{s \text{ tribbles}}
   \]
   leads to \( \frac{qs}{t} = ? \)

Notice you do not need to know what quibbles & tribbles are.
Commutative and Associative

Commutative/associative: The concept of Commutative is recorded as $A+B=B+A$ but it is enlightening to see the concept analogously or the point is missed.

Consider the action of putting on your hat and your coat. If you put on your hat then put on your coat OR put on your coat and then put on your hat the outcome is unaffected by this choice. However if your action involves putting on your shoes and socks then the outcome is definitely different. If you put on your shoes then cover them with your socks the result is quite different from when you put on your socks then your shoes. So commutativity is about order and whether you can change it under an action, without affecting the outcome. So when the action is not order sensitive this means it is commutative. Order sensitive says not commutative. Since addition and multiplication enjoy this freedom of order, they are analogous to your hat and coat whereas subtraction and division are order sensitive they are analogous to your shoes and socks.

Next is associative which is the grouping issue for sure, but the change in the placement of the parenthesis is about emphasis.

It is recorded as $(A+B)+C=A+(B+C)$ but is best seen through an analogy.

Consider the following to clarify what it means to be NOT associative.

(light green) bucket VS light(green bucket) The first says it is a bucket light in the color of green but the second says it is a green bucket that is not heavy to carry.

(high school) student VS high(school) student also demonstrates how emphasis effects outcome. The first is a highschool student in 9-12th grades(USA) while the other is a HIGH school student who is high on something(hopefully math 😉)

Associative which is the grouping issue for sure. Changing the placement of the parenthesis is about emphasis and whether it effects the interpretation.

So putting these to work sees: You can group efficiently when adding or multiplying.

\[ \begin{array}{cccc}
4 & 8 & 2 & 9 \\
5 & 2 & + & 1 \\
\hline
1 & 4 & 6 & 8 \ \\
3 & 1 & 4 & 6 \\
\end{array} \]

\[ \begin{array}{cccc}
+ & 1 & 1 & \text{see that } 8+2 \text{ and } 9+1 \text{ are 10 so the first column adds to 20 so carry the ‘2’ says you carry } 2 \text{ tens. Then in the second column } 5+4+1 \text{ is 10 as well so the sum of the second columns(tens) is 14} \\
& & & 6 \ \\
\hline
& & 6 & 8 \ \\
& & 1 & 4 \\
\end{array} \]

If you have 14 X 683 you can reverse it and calculate 683 X 14 instead. X 14 VS X 683

The identity for an action(if there is one) is the element that causes no change to anyone. For addition this is zero and for multiplication it’s one. I think of it as ‘HOME” for the action.

Inverses are pairs of entities that assist each other home for the action.(back to the identity) so inverse pairs(when you have them) cluster around the identity.

So under addition inverses are opposites like -3 & 3, 4 & -4, $\frac{5}{6th}$ & $\frac{5}{6th}$.

Under multiplication inverses are reciprocals like $3 & \frac{1}{3rd}$, $\frac{5}{6th}$ & $\frac{5}{6th}$, -4 & $\frac{1}{4th}$.

Can think of the identity as the center of gravity for the system under the action. Remember if there’s no identity then the inverses are moot.
Division by Zero
The concept of division is grounded in the process of answering the question: "How much does each person get?" So with this in mind, 12 divided by 6 asks: out of 12 things how many will each of the 6 people get? The answer is 2 things.

\[
\frac{12 \text{ things}}{2 \text{ people}} \quad \text{says the numerator is the things being distributed and the denominator is the people receiving the things.}
\]

With this understanding now consider the following.

\[
\frac{0 \text{ things}}{12 \text{ people}} \quad \text{vs} \quad \frac{12 \text{ things}}{0 \text{ people}}
\]

In the first representation you have zero things to be divided among 12 people therefore each person gets nothing. Critical to understand that the division question was addressed.

In the second representation you have 12 things to be divided among no people. This means "nobody is home" so the division question collapses and goes unaddressed since you cannot answer the question: "how much did each person get?" This is why the outcome is undefined because nobody is home to receive the things being distributed.

Later algebraically this can be used to explain that when denominators go to zero (collapse), this identifies a place in the path (graph) where a discontinuity occurs. (undefined)

\[
\text{Removable discontinuities are ones that in a sense can be repaired by plugging the hole with a single point. The right hand lower corner line graph has a removable discontinuity. These are where the limit exists but are not equal. Not removable means that the repair cannot be accomplished by a single point. Jump( where the graph jumps to a different path) discontinuities}
\]
and vertical asymptotes cause these. The other graphs above display these. These are where the limit does not exist.
**Evolution of Numbers**

The development of numbers has an interesting history. The **naturals** are called the counting numbers because they were used to count objects. Keep in mind that the wealth at this time was livestock so they counted sheep, cows, goats, etc. These you can see as stepping stones going east from 1. This means if you are a very small critter walking along the numberline you’d better be careful to jump securely from stepping stone to stepping stone or off into oblivion you will fall. Zero is not in this set and therefore the **whole** numbers grew out of the need to express that I had no wealth, ie NOTHING. These are stepping stones going east from zero.

If you stayed in the confines of this set, you would not be able to borrow anything. Hence the **integers** evolved. These are stepping stones east and west without end. Thinking of these sets as stepping stones, says that if you are walking along this path (the numberline) while travelling through these sets and you miss a stepping stone then you fall into oblivion never to be seen again. Next at the party, are the **rationals**: \( \frac{n}{d} \) which are formally defined as a ratios between any two integers, but \( d \) is not 0. Think of this process as a game of picking an integer from the integer bag and before you pick the second choice you must remove the troublemaker (zero) from the bag. Remember division by zero is undefined since division always asks the question: "How much does each person get?" So if we have \( \frac{6}{0} \) this says 6 things to be divided among no people (nobody is home) so the division collapses and we cannot answer the question that division addresses.

These are the pebbles around the stepping stones yet there are still spaces between these so once again if you miss a stepping stone or pebble it's off to oblivion again.

It is not until the **irrationals** arrive that the numberline path is dense which means as you walk along the path there are no fears of falling through. The irrationals come from the points that lie between any two perfect roots, like between \( \sqrt{25} \) which is 5 and \( \sqrt{36} \) which is 6. When the irrationals arrive they are the sand and mud around the stepping stones and pebbles. The blending of the rationals and the irrationals creates the **REALS (the real numberline)** which has no spaces through which to fall.

Two sets are equivalent this means they contain the same AMOUNT of information. So if one set has 5 elements \( \{1,2,3,4,5\} \) and \( \{a,b,c,d,e\} \) these are equivalent. Sets are equal if they contain
EXACTLY the same information. Turns out that the naturals, whole, integers, and rationals are all equivalent.
Expressions Equations Literal

It is critical that you clearly see the differences between expressions and equations. Consider the following:

<table>
<thead>
<tr>
<th>EXPRESSIONS</th>
<th>EQUATIONS</th>
</tr>
</thead>
<tbody>
<tr>
<td>2x</td>
<td>2x=10</td>
</tr>
<tr>
<td>2x + 5</td>
<td>2x + 5 = 15</td>
</tr>
<tr>
<td>2(x + 5) - 18</td>
<td>2(x + 5) = 18</td>
</tr>
<tr>
<td>3(x + 1) - 5(x + 2)</td>
<td>3(x + 1) = -5(x + 2)</td>
</tr>
<tr>
<td>x^2 + 10x - 24</td>
<td>x^2 + 10x - 24 = 0</td>
</tr>
</tbody>
</table>

The point is that there is a monumental difference between an expression and an equation. An expression is as best simplifiable whereas and equation is possibly solvable. The numbers which come out of an equation are related to the geometry behind the equation whereas an expression has no geometry behind it. You can think of the skills developed in expressions as practice for the game whereas these skills put to work in equations are the game. The equal sign is the expression is simply a prompter to simplify if you can.

2x is an expression, therefore owns no place in because it says space because it says "double me" and there is no way to find out who 'me' actually is. Whereas 2x = 10 yields a result of 5 “double me yielded 10” which is a point on Numberline 5 units from zero. <---

Similarly given x^2 + 10x - 24 is an expression and equation. Factoring is not the issue factorizing is the issue. which you get the numerical answers which locate the x-axis hits for the equal to zero.

Given an expression you can evaluate it for particular values. I see this as a recipe and the ingredients. Any change in the recipe or the ingredients will alter the outcome produced. Evaluate x^2 – 3xy for x = -1 & y =2 sees: (-1)^2 –3(-1)(2) is 1+6 for 7. Evaluate 3x^2 +5xy –y for x= -2 & y = 3 sees: 3(-2)^2 +5(-2)(3) –3 is 12 –30 –3 for -21 Evaluate 7xy – x^2 for x =1 & y =4 sees: 7(1)(4) – 1^2 is 28 -1 for 27. NOTE: -1^2 ≠ (-1)^2 and globally - x^2 ≠ x^2 -1^2 says square first then negate whereas (-1)^2 says negate first then square. Two different recipes. If you take an expression and set it equal to a value you now have an equation. 2x set equal to 10 finds x = 5.

To solve equations in one variable 1st degree there are stages to be followed. Addition is reversed by subtraction and visa versa. Multiplication is reversed by division and visa versa. You can think of this as a set of directions to go somewhere. If the going directions involve
addition it is like making a right to get there so you will make a left (subtract) to get home. So \( x + 5 = 13 \) seeks to find somebody added to 5 that yielded 13. The result is 8.

Whereas \( x - 5 = 13 \) leads to 18 since you seek somebody who loses 5 and lands at 13.

For every addition there are two subtractions which means \( 5 + 13 = 18 \) also says \( 18 - 13 = 5 \) AND \( 18 - 5 = 13 \). Similarly every multiplication (except by zero) has two divisions behind it. So \( 5 \times 8 = 40 \) also says \( 40 \div 5 = 8 \) AND \( 40 \div 8 = 5 \). So \( 5 \times x = 40 \) seeks to find 8. (5 times somebody is 40) whereas \( 8 \times x = 40 \) seeks to find 5. (8 times somebody is 40) These simple ones would mean you live in my neighborhood. When the equation becomes more complicated you are in my town, then in my state, then in my country etc… the more directions it takes to get there and home the further away you are.

Now if there is more than one connection then they must be reversed in the order in which they were given in forward motion. So \( 2 \times x + 6 = 12 \) which in language says “double me plus 6 is 12. So reversing this subtracts 6 from 12 then divide by 2. In stages we see \( 2 \times x + 6 = 12 \) sees \( 2 \times x = 6 \) which says 2 times X is 6 so X finds 3.

If there are any fractional controls then multiply the entire equation by the LCD to clear the fractions.

1st distribute.
2nd clean up any mess on either side.
3rd Plant the focus(variable)

Lastly: 4th remove any remaining connectors away from the focus.

In my town:

Stage 4 only:

\[ 2x + 6 = 12 \]

VS

\[ 2(x + 6) = 12 \] distribute to see

Subtract 6 to see:

\[ 2x = 6 \] so \( x = 3 \)

\[ 2x = 0 \] says 2 times somebody is zero so \( x = 0 \)

In my state:

Stage 1, 2 and 4

\[ 2(x - 5) - 7 = 1 \] distribute to see

\[ 2(x + 5) = x + 1 \] distribute to see

\[ 2x - 10 - 7 = 1 \] clean up the mess

\[ 2x + 10 = x + 1 \] Plant the focus(variable) by subtracting X

\[ 2x - 17 = 1 \] add 17 to see

\[ 2x = 18 \] so \( x = 9 \)

\[ x + 10 = 1 \] subtract 10 to see

\[ x = -9 \]

In my country:

Stage 1,2,3 and 4

\[ 5(x + 2) - 7 = 13 - 3(x - 2) \] distribute to see

\[ 5x + 10 - 7 = 13 - 3x + 6 \] clean up the mess to see

\[ 5x + 3 = 19 - 3x \] add 3x to plant the focus

\[ 8x + 3 = 19 \] subtract 3 to see

\[ 8x = 16 \] so \( x = 2 \)

Fractionally controlled equations:

\[ \frac{x}{5} + 1 = \frac{x}{3} \] multiply by 15 (LCD)

Whenever there are fractionals involved, first clear them by multiplying by the LCD.

\[ 3x + 15 = 5x \] subtract X to see

\[ 15 = 2x \] so \( x = \frac{15}{2} \)
Once the concept of solving equations is clear then develop the skill to solve literal equations.

\[ 2x + 6 = 12 \text{ leads to } \]
\[ 2x = 6 \]
\[ x = 3 \]

I see this difference as when you can actually solve for ‘x’, you call my house and I answer the phone. I am answering all the questions directly and am able to process numerically. The second literal situation is when someone calls my house and my husband takes a message recording the questions being asked. Number crunching does not happen but there is a record of the conversation.

**Literal solving** uses the same procedure as actual solving does.

\[ 2(x + 6) = 12 \text{ leads to } \]
\[ 2x + 12 = 12 \]
\[ 2x = 0 \text{ says } x = 0 \]

\[ 5x + 6 = 2x + 1 \text{ subtracting } 2x \text{ sees } \]
\[ 3x + 6 = 1 \text{ subtracting 6 sees } \]
\[ 3x = -5 \]
\[ x = \frac{-5}{3} \]

So \( x = \frac{a - c}{a - c} \) to uncover the coefficient, factor out the ‘x’ to reveal who to divide by.

Whenever there is a “split focus” this process of factoring will be needed as the tool to uncover the coefficient.

\[ \frac{x}{a} + b = \frac{x}{c} \text{ multiply by ac (LCD) } \]
\[ cX + acb = aX \]
\[ acb = aX - cX \]
\[ acb = (a-c)X \]
\[ \text{So } X = \frac{acb}{a-c} \]
Factoring

Factoring means you are looking for the parts from which the polynomial came. The first type of factoring, **common kind**, scans for a common factor found in each term, & comes in 3 flavors.

Could be simply a number: 3X^2 +9Y sees 3(X^2 + 3Y)
Could be simply a variable: 3X^2 + 8XY sees X(3X +8Y)
Could be a combo of both variable & number: 3X^2 + 9XY sees 3X(X +3Y)

Recognize that common factoring is simply the reversal of some distribution.

**Difference if two perfect squares**: A^2 – B^2 factors into: (A+B)(A– B). Think of A & B as the ingredients.

So the recipe verbalizes as: (sum of the parts)(difference between of the parts).

**Example**:  For 25X^2 – Y^2 the parts(ingredients) are: 5X & Y so the factors are: (5X+Y)(5X–Y )

**Special cubics** look like A^3 + B^3 or A^3 – B^3

- The sum of two cubes: A^3 + B^3 factors into: (A+B)(A^2 – AB + B^2)
- The difference of two cubes: A^3 – B^3 factors into (A–B)(A^2 + AB + B^2)
- These can be compressed into: A^3 + B^3 = (A±B)(A^2 ± AB + B^2) which verbalizes as:
  - (lift the cubes) [(1st ingredient)^2 change of sign (1st ingredient )•(2nd ingredient) + (2nd ingredient)^2 ]

**Example**: 8X^3 + 27 has ingredients of 2X & 3 so the factors are (2X+3)(4X^2–6X + 9)

If there is a common factor within, remove it first before a secondary factoring may occur.

**Example**: 16X^2 – 36 first sees 4(4X^2–9) which then factors by difference of squares as: 4(2X+3)(2X–3)

**Factor by grouping** requires 4 terms or more to be applicable. 3X^2 + 2X + 6XY + 4Y

First separate the terms as 3X^2 + 2X + 6XY + 4Y to see that the first bunch factors as: X(3X+2) and the second bunch factors as 2Y(3X+2) and notice there’s a common factor of 3X+2.

3X^2 + 2X + 6XY + 4Y separates as: X(3X+2) + 2Y(3X+2) leading to:

(3X+2)(X+2Y) If you multiply this out you will regain the original 4 terms. If not something’s wrong.

No rearrangement of the terms changes the outcome. 3X^2 +6XY + 2X + 4Y leads to 3X(X+2Y) + 2(X+2Y) which leads to (X+2Y) (3X+2) which is the same as grouping above.

If the common factor does not surface then it’s not factorable. Consider: 3X^2 + 2X + 6XY + 3Y

X(3X+2) + 3Y(2X+1) so since the factor is not common you cannot continue the process.

Sometimes the expression need tweaking. 3X^2 – 2X – 6XY + 4Y. Notice the first bunch is a flow + to – while the second bunch is a flow of – to + . So it needs tweaking by also factoring out a negative in the second bunch. This sees X(3X–2) - 2Y(3X–2) which leads to (3X–2) (X–2Y)

With 4 terms it is generally a 2 by 2 form of grouping. But it might be a 3 by 1 form.
Example:  $X^2 - 8X + 16 - 25Y^2$. Grouping the 1st three terms it factors into $(X-4)(X-4)$ which is $(X-4)^2$ so you have $(X-4)^2 - 25Y^2$ which is the difference of two perfect squares with ingredients of $(X-4)$ & $5Y$ so you have $(X-4 + 5Y)(X-4 - 5Y)$. 
Lines intersect when the slopes are not the same. The question is where do they intersect (where do the signals agree)? You can graph the lines on the same axis and “hope” you can read the point of intersection but if this point is fractional it will be difficult if not impossible to read it off a graph.

If you seek the point of intersection the you can set the equations (signals) equal to each other and this tells them to tell you where they agree.

So given \( y = 3x - 5 \) and \( y = -x - 1 \) setting them equal to each other reveals: \( 3x - 5 = -x - 1 \) which solves to find \( x = 1 \). Now using this value in either equation reveals \( y = -x - 1 \) that sees \( y = -(1) - 1 \) for \(- 2\). So the point of intersection is \((1, -2)\) as seen in the graph.

Given: \( y = -2x - 3 \) and \( y = \frac{1}{2}x + 2 \) setting them equal to each other reveals \(-2x - 3 = \frac{1}{2}x + 2\) Now multiply by 2 to clear the fractional control to see \(-4x - 6 = x + 4\) which solves to see \(-5x = 10\) so \(x = -2\)

Now take \( x = -2 \) in either equation (they agree here) to find the value of \( y \).

\( x = -2 \) in \( y = -2x - 3 \) sees \(-2(-2) - 3\) which is 1. So the point of intersection (agreement) is \((-2, 1)\).

Now the next one is not ready to set equal to each other so the tool used for this one is **Gaussian elimination**. Gaussian elimination is a process created by Gauss (Carl Friedrich) 17-1800’s by which you have or create same size opposite sign on one of the locations (x OR y) so that when you add the equations to each other this eliminates one location allowing the other to tell you its value. Then take the found value and use either equation (they agree at this moment in time) to recover the other value. These numbers constitute the point of agreement between the signals (point of intersection).
Imagine you have been invited to a party and asked to bring a dessert. Then you can choose whatever you want to bring but once you decide, this determines the recipe to cook it up. But if you buy the dessert then it is ready for you. These are called “readable to add”. The system is either ready to add or needs some cooking up to create same size with opposite sign for Gauss’s tool to accomplish the goal.

If it needs some cooking up then the numbers determine the recipe used.

Example:

\[ x - y = -1 \]
\[ 3x + y = 9 \]

Now this one is **ready to add on** ‘y’ so adding these reveals:

\[ 4x = 8 \quad \text{so } x = 2 \]

then using \( x - y = -1 \) with \( x = 2 \). Putting this in \( x - y = -1 \), sees \( 2 - y = -1 \) which says \( y = 3 \) so the point of intersection (where these signals agree) is \( (2,3) \) as seen in the picture above.

Example:

The equation for green line is \( x + 2y = 6 \), the pink one is \( x + y = 2 \).

Since same sign opposite sign is not given so we need to create it. **Choosing** to eliminate ‘x’ you multiply the second equation by -1 to create \( x \) against \( -x \). It is not ready to add as given.

\[ x + 2y = 6 \quad \text{stays the same} \quad x + 2y = 6 \]
\[ x + y = 2 \quad -1(x + y = 2) \quad \text{sees} \quad x - y = -2 \]

\[ y = 4 \quad \text{using this in } x + y = 2 \text{ reveals } 4 + y = 2 \rightarrow y = -2 \quad \text{so } (-2,4) \text{ is where they agree(point of intersection) seen in the graph above.} \]
Example:

Sometimes an equation needs adjustment by moving the furniture around which does not change the path of the equation. Adjust $5x + 6y - 8 = 0$ into $5x + 6y = 8$ by moving the 8 (furniture around).

And $2x + y + 1 = 0$ into $2x + y = -1$. Now the system needs some cooking up for elimination to occur.

Choosing to eliminate ‘y’ you want to cook up: $6y$ & $-6y$. So multiply the second signal by -6 to see:

$5x + 6y = 8$

$2x - y = -1$ becomes $-12x - 6y = 6$ multiplied the second signal by -6

which reveals: $-7x = 14$ so $x = -2$. Using this in $2x + y = -1$ leads to $2(-2) + y = -1$

$\rightarrow -4 + y = -1$ so $y = 3$ so $(-2, 3)$ is where these lines intersect (where the signals agree in graph above).

You can determine that a system is parallel or actually the same line by observing the following.

$-3x + 2y = -1$

$6x - 4y = 7$ Notice that the second equation is a multiple of the first on the left but not on the right so this means they have the same slope and are therefore parallel. If you apply Gaussian elimination you see it collapse. Multiplying the top equation by 2 sees:

$2(-3x + 2y = -1)$

$-6x + 4y = -2$

$6x - 4y = 7$ $6x - 4y = 7$ this leads to $0 = 5$ which is false which says there’s no point of intersection therefore parallel.

$-3x + 2y = -1$

$6x - 4y = 2$ Notice that the lower equation is an EXACT multiple of the other (by 2), so these are the same line. When you add 2 times the top equation to the lower equation this yields $0 = 0$ which says always true with each other therefore these signals are one in the same line.

Now imagine finding the point of intersection between three or more lines or other structures. If you seek to find points of intersection between two structures setting the signals equal to each other tells them to tell you where they agree. Here Gaussian elimination may need to be applied repeatedly.
Here (1,1) is the point of agreement for all 3 lines.

Gaussian elimination can be used on structures that are the same. So if you wanted to find the point(s) of intersection between two circles then Gaussian elimination will process this.

\[ X^2 + Y^2 = 9 \]

\[ X^2 + Y^2 = 25 \]

multiply the top equation by -1 and add to the lower equation yields \(0 = 16\) which says these circles do not intersect. Both are centered at (0,0) one has radius 3 and the other radius 5 so these will not intersect.

\[ r = 3 \quad \& \quad R = 5 \]

If the structures are not the same like a line and a circle, Gaussian elimination is not a useful tool.

Substitution is the tool to be used. \(Y = X + 1\) and \(X^2 + Y^2 = 25\) Substituting \(X + 1\) into \(X^2 + Y^2 = 25\) sees:

\[ X^2 + (X+1)^2 = 25 \]

and expanding this finds \(X^2 + X^2 + 2X + 1 = 25\) for \(2X^2 + 2X - 24 = 0\) which sees \((X+4)(X-3)=0\)
for $X = -4$ and $X = 3$ so the points of intersection, using $Y = X + 1$ find: (-4, -3) and (3, 4)

Setting these equal to each other sees:

$$-x + 1 = x^2 - 1 \rightarrow 0 = x^2 + x - 2 \rightarrow$$

$$0 = (x - 1)(x + 2)$$

which says $x = 1$, $x = -2$ then recover the $y$ values using $y = -x + 1$ to see: (-2, 3)(1, 0) as the points of intersection.

To find a point of agreement setting the signals equal to each other, tells them to tell you where they agree.

In 3-D these are the ways that planes can intersect but the only way there is a single point of intersection between three planes is when they interest in the corner of a room. The ceiling and two walls will intersect in the corner of the room. Similarly when the two walls and the floor meet they intersect in a corner. $3x + y - 2z = 5$ generates a plane not a line.

These figures do not have a single point of intersection (agreement). These have a single point of intersection between ALL the planes.
1. \(x + y - 2z = 2\)
2. \(x - y + z = 6\)
3. \(2x - 2y - 3z = 2\)

**3-D example:** This system represents three planes in space so you are interested in finding if they intersect where it happens.

So Gaussian elimination will need to be performed twice. The first application will compress this system to a 2 by 2 then a repeat Gaussian application will solve for a position then feeding this back into the 2 by 2 recovers the second position then using these two values in the 3 by 3 recovers the third value.

So choosing to eliminate \(y\) we have:

\[
\begin{align*}
y & & 2y \\
-y & & -2y
\end{align*}
\]

and then to create: \(-2y\) so using the top equation as the one to bank against the other 2 we see:

\[
\begin{align*}
x + y - 2z &= 2 \\
x - y + z &= 6 \\
2x - z &= 8
\end{align*}
\]

You can choose any of the 3 equations to be the bank equation. Here I used the middle one for the job.

Now take this 2 by 2 system and solve for either \(x\) or \(z\).

\[
\begin{align*}
2x - z &= 8 \\
4x - 7z &= 6
\end{align*}
\]

So multiply the top equation by \(-2\) to see: \(4x - 7z = 6\)

\[
\begin{align*}
-4x + 2z &= -16 \\
-5z &= -10 \\
\text{so } z &= 2
\end{align*}
\]

Now take \(z = 2\) and use either equation from the 2 by 2 system to recover the value of \(x\).

\[
\begin{align*}
2x - z &= 8 \\
x - 2y - 3z &= 5
\end{align*}
\]

with \(z = 2\) and \(x = 5\) sees \(2x - 2 = 8\) so \(2x = 10\) says \(x = 5\)

Now take both \(z = 2\) and \(x = 5\) in any of the 3 by 3 controls to get the value of \(y\).

\[
\begin{align*}
x - y + z &= 6 \\
x - 2y - 3z &= 5
\end{align*}
\]

with \(z = 2\) and \(x = 5\) sees: \(5 - y + 2 = 6\) so \(y = 1\) This says the point of intersection for these planes is \((5, 1, 2)\) visually 5 east, 1 north and 2 up

All three of these planes are parallel since the coefficient controls are the same but the outcomes are different. You have contradiction here since the same coefficient controls can’t go to different places simultaneously.

\[
\begin{align*}
x - 2y - 3z &= 5 \\
x - 2y - 3z &= 7 \\
x - 2y - 3z &= 9
\end{align*}
\]

Now imagine trying to find points of intersection here. 😊
**Graphing Lines**

I often connect algebra and geometry through the following.

<table>
<thead>
<tr>
<th>Algebra</th>
<th>Geometry</th>
</tr>
</thead>
<tbody>
<tr>
<td>Equation</td>
<td>Graph</td>
</tr>
<tr>
<td>Signal</td>
<td>Picture from the points the signal collects</td>
</tr>
</tbody>
</table>

You can tell it’s a line by observing that the signal(equation) has NO powers other than 1st degree and NO cross terms which are built upon products like \( xy \), \( 5x^2y \), \( 3xy^2 \). Powers other than 1 and cross terms are the bending forces in space so if the equation has no bending forces then it cannot curve ergo a line.

### Graphing lines:

\[ Y = \frac{1}{2}X + 1 \]

For \( Y = \frac{1}{2}X + 1 \) the tool to use is **Slope eyes**

by recognizing that the \( \frac{1}{2} \) is the slope and recognize this is the equation of a line.

When

the 1 is the y-axis hit which is (0,1). Slope info and measures the steepness (speed) of the line info.

which shows how steep or not so steep it sees:

will move through space. The slope of \( \frac{1}{2} \) or 2 will move through space. The slope of \( \frac{1}{2} \) or 2

and when

“1 to 2” says for every 1 up I go, 1 also go sees: \(-X = 2\) for 2 right. So this line hits the y-axis at 1 then these you moves at a rate of speed of 1 to 2.

**Slope eyes** is the best tool when y stands alone clustered

It’s like recognizing the difference between when you need a hammer VS a screw driver.

\[ -X + 2 \, Y = 2 \]

For \(-X + 2 \, Y = 2\) the tool to use is **Zero eyes**

which homes in on the x and y axis hits.

When

you let ‘x’ be zero this scans the y-axis for

when you let ‘y’ be zero this scans the x-axis

So for \(-X + 2 \, Y = 2\) when ‘x’ is zero this

\[ -X = 2 \] for \((-2,0)\) as the x-axis hit. Using

can locate the position for the line.

**Zero eyes** the best tool when \( x \) & \( y \) are clustered
Elementary Algebra (041) Review

Start at (0,1) move up 1 right 2

(-2,0)

Now if the line’s equation has a common factor then you can divide by it to simplify.

So for $6x + 2y = 10$ divided by 2 is $3x + y = 5$

divide by 2 is

so through zero eyes see: $(0,5)$ and $(\frac{5}{3}, 0)$

The second one is $y = 3x + 2$ so these are parallel.

There are two values in a line that can be fixed. General form of a line: $y = mx + b$ slope = $m$, y-axis hit = $b$

Here are the slopes as they change. When ‘b’ is fixed while ‘m’ is free to roam, they all intersect on the y-axis. Here $b = 0$ Notice that when slope is
negative the lines fall whereas when the slope is positive the lines rises. If ‘b’ is changed to 5 then this entire system moves up the y-axis to (0,5) and if ‘b’ is changed to -5 then this entire system moves down the y-axis to (0,-5) . When slopes are the same the lines will be parallel. So when ‘m’ is fixed while ‘b’ is free to roam, this causes parallel systems. So Y=2X-3 and Y=2X+5 are parallel since both have slope 2.

If the slopes a negative reciprocals of each other like $\frac{-3}{2}$ & $\frac{2}{3}$, then the lines will be perpendicular. So Y= $\frac{-3}{2}$ X+4 and Y= $\frac{2}{3}$ X +1 are perpendicular.
**Integer Reactions**

The number line eyes were not the first interpretation relative to the actions using **negative** and **positive** numbers.

The first interpretation were understanding losses and gains then the numberline gives a geometric set of eyes.

The negative was used to record "I borrowed a cow". Keep in mind the early measurement of wealth was livestock not money.

To understand the reactions of positive and negative numbers, think about positive as money you have(gains) VS negatives as money you owe(losses).

So under **addition** there are 4 possibilities:

1: \(3 + 8\) says a gain of 3 followed by a gain of 8 which yields a **TOTAL** gain of 11.

2: \(-3 + (-8)\) says a loss of 3 followed by a loss of 8 which yields a **TOTAL** loss of 11: -11.

3: \(-3 + 8\) says a loss of 3 followed by a gain of 8 which yields a **NET** gain of 5.

4: \(3 + (-8)\) says a gain of 3 followed by a loss of 8 which yields a **NET** loss of 5. -5

Notice that when you have a gain followed by a gain this means you made money and did not spend any in between hence a **TOTAL** gain.

Similarly when you have a loss followed by a loss this means you spent money and made none in between hence a **TOTAL** loss.

However when you make money and spend as well, it becomes dependent upon which is larger. (size sensitive).

This sensitivity applies in the 3rd and 4th examples above but not in the 1st and 2nd

To understand **subtraction** of negative/positive numbers relate it to the insights under addition.

First of all, subtraction is a loss since if you take it away from me I have lost it.

Secondly, consider the outcome caused by \(-(-8)\). This represents the cancellation of a debt of 8. -8 says you owe someone $8. But \(-(-8)\) says they cancel the debt which becomes a gain to you.

There are again 4 possibilities:

1: \(8 - 3\) is 5 since it says: a gain of 8 followed by a loss of 3 yielding a **gain** of 5. This is a simpler form of #3 in addition land.

2: \(3 - 8\) is -5 since it says: a gain of 3 followed by a loss of 8 yielding a **loss** of 5. This is a simpler form of #4 in addition land.

3: \(-3 - 8\) is -11 since it says: a loss of 3 followed by a loss of 8 yielding a **TOTAL** loss of 11. This is a simpler form of #2 in addition land.

4: \(3 - (-8)\) is 11 since it says: a gain of 3 followed by the cancellation of a debt of 8 so it becomes 3+8 yielding a **TOTAL** gain of 11. This is a simpler form of #1 in addition land.

So the 4 possibilities under subtraction are actually the 4 possibilities from addition through different eyes.

Since **multiplication** is sped up addition, this means that 3 times 8 is actually 8 added three times. \((3)(8)\) is \(8 + 8 + 8\) which is 3 gains of 8 which yields 24
There are 4 possibilities:
1: (3)(8) is 24 since it says you have 3 gains of 8 which yields a TOTAL gain of 24.
2: (3)(-8) is -24 since it says you have 3 losses of 8 which yields a TOTAL loss of 24.
3: (-3)(8) is -24 since it says you have the loss of 3 eights which yields a TOTAL loss of 24.
4: (-3)(-8) is 24 since it says you have the cancellation of the debt of 3 eights yielding a TOTAL gain of 24.

This last possibility builds upon the cancellation of a single debt, -(3)(-8) as seen in subtraction. It is an upgrade to more than one cancellation of debt.

Notice that multiplication is NOT size sensitive which means whichever number is bigger does not drive the outcome relative to being a gain or a loss.

Whereas addition is very size sensitive when combining losses and gains. Under addition when the gain outweighed the loss you made more than you spent. Whereas when the loss outweighs the gain you owe since you spent more than you made.

Although we can "look back" at multiplication to justify division, the process of division actually says the following.
The concept of division is grounded in answering the question: "How much does each person get?" So 24 things divided among 8 people says that each person gets 3 things.

There are 4 possibilities:
1: \( \frac{24}{8} \) says 24 gains divided among 8 people, so each person gains 3 things.
2: \( \frac{-24}{8} \) says the loss of 24 things divided among 8 people, so each person owes 3 (loss of 3) for -3.
3: \( \frac{24}{-8} \) Notice the negative is on the people, which says: 8 people in debt by 24 things so each person owes 3 (loss of 3) for -3.
4: \( \frac{-24}{-8} \) is the same as \( \frac{(-24)}{8} \) says the cancellation of a debt of 24 things for 8 people so each person gains 3.

#4 once again builds upon the cancellation of a debt from subtraction and expands it to the cancellation of more than a single debt.

Once this is clear the numberline(geomety eyes) makes sense. The numberline interpretation is needed for later analysis. The loss/gain eyes preceded the numberline by about 100 years. The numberline can be seen as losses & gains in a football game.
Like Terms Exponents

Like terms is really about same sizes or objects. If you have 3 fives and 6 fives, how many fives do you have? 9 fives right? This is 3f + 6f which is 9f algebraically.

Similarly if you have 4 tens and 3 tens then you have 7 tens which is algebraically 4t + 3t which is 7t algebraically.

However if you have 3 fives and 6 tens, then you cannot make 9(five-tens) since 3f + 6t cannot create 9 of anything.

Related to dollar bills the point is that:

4($10-bills)+3($10-bills) yields 7($10-bills) which is 4t+3t or 7t whereas 3($5-bills) + 6($10-bills) cannot make 9($15-bills) which is 3f + 6t algebraically.

Analogously, think geometrically if you have: 3 boxes and 5 boxes then you have 8 boxes. This is 3B + 5B = 8B algebraically.

Similarly if you have 4 spheres and 7 spheres then you have 11 spheres. This is 4S + 7S = 11S algebraically.

But if you have 3 boxes and 4 spheres can you make this into 7 (boxspheres)? No! So algebraically this is 3B + 4S which cannot be combined. 3 + 4 can’t create boxspheres.

This confirms the fact that in the fractional arena (in fact any arena), addition demands common denominators (sizes) because addition and subtraction are size sensitive.

Now take note of the fact that size is driven by not only the variable structure but also the power structure. So 3x^2y is not combinable under addition with 8xy^2 since they are not the same size, but 3x^2y and 8x^2y will produce 11x^2y algebraically.

Multiplying is not size sensitive. So 3t times 6t creates 18t^2. This occurs because (3:tens) times (6:tens) produces 18:hundreds since 3 tens is 30 and 6 tens is 60 therefore 30 times 60 produces 1800 i.e. 18:hundreds. So algebraically this is (3t)(6t) which yields 18t^2.

This confirms the fact that in the fractional arena, multiplication is NOT size sensitive which is why under multiplication of fractions, common denominators (sizes) are not warranted. Notice that when multiplying the same base the powers react by adding. So (3x^4)(4x^3) is 12x^7 but you cannot add:3x^4&4x^3.

When you have something raised to yet another power the powers react by multiplying. (3x^4)^2 leads to 9x^8. Now it is critical that you pay attention to how far the arm of jurisdiction of the power goes. 3(x^4)^2 is 3x^8 since the square has local jurisdiction over x but not 3.

Whereas (3x^4)^2 is 9x^8 since the square has global control over all parts involved. So 2(x^3y^4)^2 is 2x^6y^8 is locally controlled.

I see locally controlled as state laws and globally controlled as federal laws.

Since division is the reverse of multiplication then the powers will react by subtracting. It is always the numerator’s exponent minus the denominator’s exponent. x^{15}/x^5 leads to x^{10} but x^5/x^{15} leads to x^{-10} which is 1/x^{10} since the denominator is the heavier weight. Notice that x^{15}/x^5 & x^5/x^{15} are reciprocals so therefore the results of x^{10} & 1/x^{10} are also reciprocals. Recognize that negative
powers have nothing to do with negativeness (in debt). Negative powers are fractionalizers (shrink me). So \(2^{-3} = \frac{1}{8}\) & \(5^{-2} = \frac{1}{25}\)

\[
\frac{3x^2y^3}{9x^5y^2} \text{ leads to } \frac{x^{2-5}y^{3-2}}{3} = \frac{y}{3x^3} \quad \text{VS} \quad \frac{9x^5y^2}{3x^2y^3} \text{ leads to } 3x^{5-2}y^{2-3} = \frac{3x^3}{y}\]

Once again you have to pay attention to whether you have a locally controlled or a globally controlled situation.

\[
\left(\frac{3x^3}{9x^2}\right)^2 \text{ is locally controlled \quad \left(\frac{3x^3}{9x^2}\right)^2 \text{ is globally controlled}
\]

\[
\frac{9x^6}{9x^2} = x^4 \quad \quad \quad \quad \frac{9x^6}{81x^4} = \frac{x^2}{9}
\]

In globally controlled situations you can clean up internally then simplify. So \(\left(\frac{3x^3}{9x^2}\right)^2\) can be reduced internally to \(\frac{x}{3} = \frac{x^2}{9}\)

In locally controlled situations reducing within is illegal. So \(\left(\frac{3x^3}{9x^2}\right)^2\) cannot reduced 3 with 9 before applying the power.

\[
\frac{(2x^3y)^3(3x^3y)^2}{(6x^7y^4)^2} = \frac{(8x^9y^6)(9x^8y^6)}{36x^{14}y^8} = 72x^{17}y^{12} = 2x^{17-14}y^{12-8} = 2x^3y^4
\]

\[
\frac{2(x^3y)^3(3x^4y^3)^2}{(6x^7y^6)^2} = \frac{2(x^9y^6)(9x^8y^6)}{36x^{14}y^8} = 18x^{17}y^{12} = 2 \quad \quad \quad \quad \frac{x^{17-14}y^{12-8}}{2} = x^3y^4
\]

\[
\frac{3(x^3y)^2}{15x^{-2}y^{-4}} \text{ leads to } 15x^{-2}y^{-4} \text{ which is } \frac{1}{5} = 5x^4
\]

Recall \(x^{-4}\) is \(\frac{1}{x^4}\) this is why \(x^4\) is in the denominator (a divisor).

You can put the skills in exponents to work and use them to simplify.

**Example:**

\[
2^{74} - 2^{71}
\]

\[
2^{74} + 2^{72}
\]

remember you can’t simply cancel these terms for to do so would be algebraically illegal and cause a severe imbalance. So to get at this you need to factor.

\[
\frac{2^{74} - 2^{71}}{1} \quad \text{then canceling } 2^{71} \text{ you get}
\]

\[
\frac{8-1}{2(4+1)} \quad \text{which is} \quad 10
\]
Example: If asked to discern which is larger? \(27^{20}\) OR \(9^{30}\) you need to see these in the SAME base:

\(27^{20}\) is \((3^3)^{20}\) which is \(3^{60}\) and \(9^{30}\) is \((3^2)^{30}\) which is also \(3^{60}\) so the answer is neither is larger.

If the structures reveal a comparison between: \(27^{19}\) OR \(9^{30}\) then you have \((3^3)^{19}\) VS \((3^2)^{30}\) for \(3^{57}\) VS \(3^{60}\) so \(3^{60}\) is larger.

Example: Remember a negative power has nothing to do with negativeness. \(2^{-3}\) is \(\frac{1}{8th}\) so \(2^{-3} + 3^{-2}\) is actually \(\frac{1}{8th} + \frac{1}{9th}\) which processes using 72nds. Now see: \(\frac{1}{8th}\) as \(\frac{9}{72nds}\) & \(\frac{1}{9th}\) as \(\frac{8}{72nds}\) for \(\frac{17}{72nds}\).

Example: If you are asked which is larger given \((2^{-3})^{15}\) VS \((2^{-4})^{11}\)
Then this sees: \(2^{-45}\) VS \(2^{-44}\) which is \(\frac{1}{2^{45}}\) \(\frac{1}{2^{44}}\) so \(\frac{1}{2^{44}}\) is larger. Recognize that \(\frac{1}{2^{45}}\) is \(\frac{1}{2}\) of \(\frac{1}{2^{44}}\).
Order of Operations Polynomials
Order of operations is a worldwide agreement by which we process so we end up at the same result mathematically. It has four steps & governs term by term not the entire structure at one time.
1) Parenthesis are precedence givers & say DO ME FIRST
2) Exponents/powers
3) Mult/division WHICHEVER COMES FIRST mult & division are equally weighted unless there’s a parenthesis giving precedence.
4) Finally add/ subtraction is all that could be left to process.

First notice that separators are addition or subtraction NOT inside a parenthesis. These separate the processing into terms and identify how many are at the party. 

There are three terms at the following party: 3; 2(4-5)² and 15÷5•3. Given: 3+2(4-5)² − 15 ÷ 5 • 3, The (4-5)² is the life of the party while the 15 ÷ 5 • 3 is at & in the party, but the 3 is at the party but not in the party for he is observing & waiting to see what everybody else will do. So as long as you follow order of operations within each term all will be well.

Pay attention to when parenthesis are involved and when they are not.

Notice that 15 ÷ 5 • 3 leads to 3•3 for 9 whereas 15 ÷(5•3) processes to 15 ÷ 15 for 1

Reduces to 3+2(-1)² − 9
Which is 3+ 2(1) –9 which is 3 + 2 − 9 = -4
Anytime you have (1)any power it’s 1 whereas if you have (-1)even it’s 1 but (-1)odd is -1.

Now remember that anytime you multiply by zero it erases whatever it sees under multiplication. ZERO times anything is zero.

So given 4+2(3-3)any power − 18 ÷ 3 •6 leads to 4 + 0−36 for -32

So if you are trying to establish a pattern the proper process of order of operation is critical.

2+9•1= 11
3+9•12= 111
4+9•123= 1,111
5+9•1234= 11,111 If you do not process correctly then you may get a different pattern (that is wrong) or no pattern at all when there is actually a pattern.
Consider : 3+9•12= If you process the addition first as 12•12 for 144 then this incorrectly reveals an incorrect outcome. By order of operations 3+9•12 sees : 3+108 for 111 correctly processed.

It is critical that order of operations is processed correctly so we end up at the same destination. The separators determine how many terms are at the party. Similarly polynomial (multi- termed) algebraic expressions are classified by how many terms are involved.

Monomials have ONE term which means NO addition or subtraction. 3x, 5x²y, 2x³y² are monomials.
Notice that when there is NO operational symbol between structures it defaults to multiplication. $3x$ says $3$ times $x$. $5x^2y$ says $5$ times $x$ squared times $y$ so if $x = -1$ and $y = 2$ then $5x^2y$ sees $5(-1)^2(2)=10$

So for this recipe and these ingredients $5x^2y$ is worth $10$. You can see expressions as recipes and the numbers as the ingredients to put into the recipe.

**Binomials** have TWO terms which means there’s one separator. $2x + 1$, $3x^2 + 5x$, $-6x^3 - 7xy$

**Trinomials** have THREE terms so there are two separators. $4x^2 - 2x + 1$, $3x^2 + 5xy - 4$, $-6x^3 - 7xy + 2y$

Next consider what can possibly happen when operating with polynomials.

A monomial ± monomial can at most produce a binomial. $3x + 5x = 8x$ but $3x^2 + 5x$ cannot be combined. Monomial ± Binomial can produce at most a trinomial. $3x^2 + 5xy - 4x^2 = -x^2 + 5xy$ but $3x^2 + 5xy - 4x^2y$ cannot be combined. Binomial ± binomial can produce at most $4$ terms.

How many terms invited to the party determines how many can possibly be in attendance but some may combine and come together.

In multiplication a monomial (monomial) produces a monomial: $(3x^2y)(5xy^2)$ is $15x^3y^3$ but in addition $3x^2y + 5xy^2$ cannot be combined because addition is size sensitive.

Monomials (binomial) will be in the binomial family. $3x^2(5x - 2) = 15x^3 - 6x^2$

Monomials (trinomials) will be in the trinomial family. $3x(5x^2 - 2x + 1) = 15x^3 - 6x^2 + 3x$

Monomial multiplication is very predictable. It causes the outcome to be in the family over which the monomial is multiplied.

(binomial)(binomial) can produce at most $4$ terms.

Recall: \[
\begin{array}{ccc}
32 & \times & 23 \\
46 & + & 73 \\
69 & \Rightarrow & 923
\end{array}
\]

which is $2$ digits times $2$ digits \[
\begin{array}{ccc}
3x + 2 & \times & 4x + 6 \\
\Rightarrow & + & 6x^2 + 9x \\
\Rightarrow & \Rightarrow & 6x^2 + 13x + 6
\end{array}
\]

So polynomial multiplication mimics digital multiplication. The faster way is to see:

\[
\begin{array}{ccc}
\checkmark & 4x \\checkmark & \downarrow \\
\checkmark & 9x \\checkmark & \downarrow \\
( 2x + 3 ) \bullet ( 3x + 2 ) \\
6x^2 \uparrow \checkmark & \Rightarrow & \checkmark \Rightarrow 6x^2 + 13x + 6
\end{array}
\]

To process: $(3x + 4)(2x^2 + x - 5)$

\[
\begin{array}{ccc}
\Rightarrow & \text{Lock in the } 3x \text{ and send it through for:} & 6x^3 + 3x^2 - 15x \\
\Rightarrow & \text{Next lock in } 4 \text{ and send it through for:} & + \phantom{x^3} 8x^2 + 4x - 20 \\
\Rightarrow & \phantom{\text{Lock in the } 3x \text{ and send it through for:}} & \Rightarrow \phantom{\text{Next lock in } 4 \text{ and send it through for:}} \phantom{6x^3 + 11x^2 - 11x - 20}
\end{array}
\]

There are two paths for division. Short VS long. $\dfrac{217}{7}$ or $7 \) $217$ VS $\dfrac{5937}{73}$ sees $73 \) $5937$

32
\[ \frac{217}{7} = \frac{210 + 7}{7} \text{ which is} \frac{210}{7} + \frac{7}{7} \text{ for } 30 + 1 = 31 \text{ This is similar to} \frac{21x^2 + 7y}{7} \text{ for } 3x^2 + y. \text{ Notice the divisor is a monomial so that means the division is splittable into parts.}

However when the divisor is \textit{not} a monomial then this will be LONG division. \( \frac{x^2 + 10x - 24}{x - 2} \) will be:

\[
\begin{array}{c}
\phantom{+} x \\
\hline
x - 2 ) x^2 + 10x - 24 \\
\phantom{+} \uparrow \uparrow
\end{array}
\]
Polynomial Division

For division to be clear, first understand that it reverses the action of multiplication. So since 2 times 6 is 12 then \( 12 \div 6 \) finds 2 and \( 12 \div 2 \) finds 6 . \( 12 \div 2 \) is also asking: “How many sets of 2 are there in 12?” while \( 12 \div 6 \) asks: “How many sets of 6 are there in 12?” It is also seen through the eyes of distributing things among people. So if I have 12 things to distribute among 6 people then each person gets 2 things.

The concept of division is grounded in the process of answering the question: "How much does each person get?" So with this in mind, 12 divided by 6 asks: out of 12 things how many will each of the 6 people get? The answer is 2 things.

\( \frac{12 \text{ things}}{2 \text{ people}} \) asks “how many things out of 12 did 2 people get?” The numerator is the things being distributed and the denominator is the people receiving the things. Whereas: \( \frac{12 \text{ things}}{6 \text{ people}} \) asks “how many things out of 12 did 6 people get?”

Now consider \( 19 \div 6 \) which will not be a whole number result. This asks: “How many sets of 6 are there in 19?”

\( 6 \div 19 \) is \( 19 \div 6 \) so since there are 3 sets of 6 in 19 the quotient is 3 with \( \frac{1}{6th} \) left. So the answer is \( 3 \frac{1}{6th} \)

You can see division as repeated subtraction. \( 19 \div 6 \) sees: \( 6 \div 19 \) which compresses to: \( 6 \div 19 \)

So division is sped up subtraction, like addition

\[-6 \]
\[-18 \] is sped up multiplication.

\[13 \]

\[-6 \]
\[-7 \] happens three times which is 18

\[-6 \]
\[-1 \]

Now if the numbers are large this repeated division will be boring and take too much time. So consider : \( 29 \div 387 \)

To subtract 29’s until there are no more would take a long time. This is why long division exists.

First, establish the first point of entry (where the division begins). For \( 29 \div 387 \) the division starts over the 38

This establishes the size of the quotient. So for this problem the output is at least 10 and can’t exceed 99.

There is 1 set of 29’s in 38

\[29 \div 387 \]

Which starts the process \( -29 \) bring down the 7 to \( -29 \) bring down 7 then think: “how many 29’s in 97?”
9 start the next step 97 “how many 30’s in 97?” or “how many 3’s in 9?” -87 which determines the next position and there is 10 left.

The result is \(\frac{13}{29\text{ths}}\).

So what would change for 29 )3876 Once again the first point of entry is over the 38 and the process will now output 3 digits

\[
\begin{array}{c}
\underline{133} \\
\underline{113}
\end{array}
\]

which means the answer is between 100 and 999 this time. 29 ) 3876 sees 29) 3876 29) 3876

There’s one set of 29’s in 38

\[
\begin{array}{c}
-29 \\
9
\end{array}
\]

97 down 7 97

87 -87 bring down 6, then

10 106 “how many 3’s in 10?”

-87

So the result is \(133\frac{19}{29\text{ths}}\) & the fractional begins when the last digit has been processed.

19

When the last step saw “how many 29’s are there in 106?” you can “think” in 30’s since 29 is closer to its upper bound of 30. So asking “how many 29’s there are in something is close to how many 30’s there are in it. You can use the closest bound as a good estimator. So if the divisor is 26,27,28,29 then “think in 30’s” while if the divisor is 21,22,23,24 then “think in 20’s”. So instead of asking “how many 29’s there are in 106?” think “how many 30’s there are in 106” but this is close to “how many 3’s there are in 10?” which is 3. If the divisor is 25 then you can use either bound but use 30 since it’s better to overplan than underplan. Later, the fractional left over will be seen as a decimal.

73 ) 5937 sees the first point of entry over the 593 so this answer will be between 10 and 99. Next to answer “how many 73’s there are in 593?” is not obvious. So think “how many 70’s there are in 593?” because 73 is closer to its lower bound. But this is close to “how many 7’s there are in 59?” and the answer is 8.

## 73 ) 5937

\[
\begin{array}{c}
8 \\
14
\end{array}
\]

next see 73 ) 5937

\[
\begin{array}{c}
82 \\
147
\end{array}
\]

then “how many 7’s in 14?”

\[
\begin{array}{c}
-584 \\
-146
\end{array}
\]

so the answer is \(82\frac{1}{73\text{rd}}\)
**Division** has two paths. **Short VS Long**. When the divisor is a monomial then this is short division.

\[
\frac{217}{7} \quad \text{or} \quad 7 \times 217 \quad \text{Short division VS} \quad \frac{5937}{73} \quad \text{sees} \quad 73 \times 5937 \quad \text{long division}
\]

\[
\frac{217}{7} = \frac{210 + 7}{7} \quad \text{which is} \quad \frac{210}{7} + \frac{7}{7} \quad \text{for} \quad 30 + 1 = 31 \quad \text{This is similar to} \quad \frac{21x^2 + 7y}{7} \quad \text{for} \quad 3x^2 + y \quad . \text{Notice the divisor is a monomial so that means the division is splittable into parts.}
\]

But when the divisor is not a monomial then this will be LONG division. Often the purpose for long division is to uncover the factors which can be used to find the zeroes of the polynomial (x-axis hits).

So now consider polynomial division for it mimics long division when the divisor is NOT a monomial.

**Example:** \(x^2 + 10x - 24\) divided by \(x-2\)

\[
\begin{array}{c|c|c}
\hline
x & x^2 + 10x - 24 & \text{which factors into} \ (x-2)(x+12) \\
\hline
\end{array}
\]

This reduces to \(x+12\)

\[
\begin{array}{c|c|c}
\hline
x-2 & x^2 + 10x - 24 & x-2 \\
\hline
\end{array}
\]

So just like in long division you look at the \(x\) in \(x-2\) to do the work.

So “What do you need to multiply ‘\(x\)’ by to get \(x^2\)”? You need ‘\(x\)’ for this job.

\[
\begin{array}{c|c|c}
\hline
x & x^2 + 10x - 24 & \text{then} \ (x-2) \\
\hline
\end{array}
\]

Then \(x(x-2)\)

\[
\begin{array}{c|c|c}
\hline
x-2 & x^2 + 10x - 24 & x-2 \\
\hline
\end{array}
\]

\[\uparrow \quad \uparrow \]

\[
\begin{array}{c|c|c}
\hline
x + 12 & \text{sees} \ -x^2 + 2x \quad \text{leading to a remainder of} \ 12x \\
\hline
12x -24 & \text{Now “What do you need to multiply} \ x \ \text{by to get} \ 12x? \ \text{“ \ You need} \ 12. \\
-12x -24 & \text{this is} \ -12x + 24 \ \text{so no remainder.} \\
0 & \text{Since} \ (x-2)(x+12) \ \text{leads back to} \ x^2 + 12x - 24 \ \text{this confirms the division is correctly done.} \\
\end{array}
\]

Notice when there is NO remainder it says it was factorable.

**Example:** Ask: ”What do you need to multiply ‘\(x\)’ by to get \(2x^2\)” ? You need 2\(x\) for this job .

\[
\begin{array}{c|c|c}
\hline
2x & 2x^2 + 10x - 15 & \text{for} \ 2x(x-2) \\
\hline
\end{array}
\]

Then \(2x(x-2)\)

\[
\begin{array}{c|c|c}
\hline
2x & 2x^2 + 10x - 15 & x-2 \\
\hline
\end{array}
\]

\[\uparrow \quad \uparrow \]

\[
\begin{array}{c|c|c}
\hline
2x + 14 & -2x^2 + 4x \quad \text{leading to a remainder of} \ 16x \\
\hline
14x -15 & \text{Now “What do you need to multiply} \ x \ \text{by to get} \ 14x?\ “ \ You need 14. \\
-14x -28 & \text{this is} \ -14x + 28 \ \text{so 13 is the remainder.} \\
13 & \text{This gets recorded as:} \ 2x+14 + \frac{13}{x-2} \ \text{To show it’s correct,} \ (x-2)(2x+14) + 13 \ \text{to recover:} \ 2x^2 + 10x - 15 \ . \\
\end{array}
\]

**Example:** Here what do you need to multiply \(2x\) by to get \(6x^2\) ? You need \(3x\).
Elementary Algebra (041) Review

\[
\begin{align*}
\text{2 } x-3 & \mid \text{ 6x}^2 + 11x -18 & \text{then } 3x(2x-3) \text{ for } 6x^2 - 9x \\
\uparrow & \uparrow & \uparrow
\end{align*}
\]

\[
\begin{align*}
& \underline{3x} \\
& 2x-3 \mid 6x^2 + 11x -18 & \text{sees } -6x^2 + 9x \text{ leading to a remainder of } 20x \\
\downarrow & \downarrow & \downarrow
\end{align*}
\]

\[
\begin{align*}
& \underline{20x - 18} & \text{Now what do you need to multiply } 2x \text{ by to get } 20x? \text{ You need } 10. \\
& \underline{-(20x -30)} & \text{this is } -20x +30 \text{ so } 12 \text{ is the remainder.} \\
& 12 & \text{So it’s recorded as } 3x + 10 + \frac{12}{2x-3} \text{ To show it’s correct, } (2x-3)(3x + 10) + 12 = 6x^2 + 11x -18 .
\end{align*}
\]

Now you have to be aware of “holes” in the polynomial. This means there’s a power missing and division needs all powers from the highest one to be present at the party. 😊

\[
4x^3 + 5x – 9 \text{ has a hole in the x}^2 \text{ position so when you set up the division you need to load: } 4x^3 + 0x^2 + 5x – 9 \text{ so all positions are accounted for.}
\]

\[
\begin{align*}
& \underline{2x^2} \\
& 2x-3 \mid 4x^3 + 0x^2 + 5x - 9 & \uparrow & \uparrow
\end{align*}
\]

\[
\begin{align*}
& \underline{2x^2 + 3x +7} \\
& 2x-3 \mid 4x^3 + 0x^2 + 5x - 9 & \uparrow & \uparrow
\end{align*}
\]

\[
\begin{align*}
& \underline{6x^2 + 5x} \& \text{so outcome is } 2x^2 + 3x + 7 + \frac{12}{2x-3} \\
& \underline{-(6x^2 -9x)} \& \text{Remember to show it’s correct you multiply: } (2x-3)(2x^2 + 3x + 7) + 12 \text{ to recover: } 4x^3 + 5x – 9
\end{align*}
\]

So if you have: \( 4x^5 + 6x^2 + 2x -7 \), this one has two holes in \( x^4 \) & \( x^3 \) so you would load: \( 4x^5 + 0x^4 + 0x^3 + 6x^2 + 2x -7 \) into the division.

\[
\begin{align*}
& \underline{2x^3 - 4x^2 +9x +13} \\
& 2x^2 + 4x -1 \mid 4x^5 + 0x^4 + 0x^3 +6x^2 + 2x - 7 & \downarrow & \downarrow & \downarrow
\end{align*}
\]

\[
\begin{align*}
& \underline{-8x^4 + 2x^3 + 6x^2} \& \text{-8x}^4 +2x^3 + 6x^2 \& \text{-8x}^4 -16x^3 -4x^2
\end{align*}
\]
Elementary Algebra (041) Review

\[ \frac{41 + 6}{2x^2 + 4x - 1} \]

\[
\begin{align*}
18x^3 + 10x^2 + 2x \\
-(18x^3 + 36x^2 - 9x) \\
-26x^2 + 11x - 7 \\
-(26x^2 + 52x - 13) \\
41x + 6 \quad \text{So the outcome is } 2x^3 - 4x^2 + 9x + 13 +
\end{align*}
\]
Rationales vs Irrationals

It is important that the concept of rational be understood from a construction standpoint as well as why they are called rationals.

They are constructed by choosing any two integers provided the second choice is NOT zero. So think of it as a game which allows you to choose any two integers from the integer bag, forming a comparison with one stipulation. For the first choice anyone can be chosen but the second choice is restricted since division by zero is trouble. (see division by zero for clarity) So before you pick the second number you have to rummage around in the integer bag and remove zero (the trouble maker in the denominator) putting it in the corner then you can pick anybody else. By the way if zero is your first choice game is over it’s 0.

The other critical insight is that when a rational is converted to the decimal world there are two personalities: 10 friendly which means the denominator is made of 2’s and/or 5’s only OR non 10 friendlies (the denominator has something besides a 2 or 5 structurally).

The decimal expansion of a 10 friendly structure will end because the denominator is made of only 2’s and/or 5’s. The decimal structure of a non 10 friendly structure will never end but eventually yields a repetitive pattern block. So this is why something like \( \frac{13}{20} \) which terminates as .65 in the decimal world. However, \( \frac{5}{6} \) (not ten friendly) yields .833333... Though it never ends it does yield a repetitive pattern block hence rational means ends or yields a repetitive pattern block.

The reverse set of eyes says that when the decimal structure either ends, like .65 or yields a repetitive pattern block like .8333... it must have a rational form from whence it came. (it’s predictable decimal structure says so).

Hence the term rational means a level of predictability as opposed to irrational says it is an unpredictable structure which never ends nor yields a repetitive pattern block structure. This was critical at the time these numbers were being investigated. Since a ten friendly structure ends it is an exact location whereas when it yields a repetitive structure like .833333... it is not exact, but can be safely rounded off with some degree of accuracy.

The irrationals however never end NOR ever yield a repetitive pattern block so rounding them off at that time was scary, since you never knew what the next digit might be and were never sure how to approximate accurately and how this would effect your calculation. Since this time, mathematical machinery has been discovered that enables us to get accuracy at virtually any level.

They have two personalities actually: 10 friendly which means the denominator is built upon 2’s and/or 5’s only or NON 10 friendly which means there’s something in the denominator besides 2 and/or 5. Like 5/6ths which has a 3 wreaking havoc for a 10 friendly system (decimals). If the rational is 10 friendly then it is projectible. For example \( \frac{4}{5} = \frac{7}{10} \) for \( \frac{8}{10} \) or .8

Similarly \( \frac{7}{20} = \frac{35}{100} \) or .35 You can predict where the 10 friendly bunch will end by
looking at the highest power on either the 2 or the 5 found in the denominator. So \( \frac{4}{5\text{ths}} \) ends in the 10ths place since the power on the 5 is 1 ie needs one place in the decimal system.

Similarly \( \frac{7}{20\text{ths}} \) ends in the 100ths place since the highest power on either the 2 or the 5 in 20 which is \((2^2 \times 5)\) is 2. So it needs two places to be accommodated in the decimal world.

So if we have \( \frac{19}{200\text{ths}} \) we will need 3 places to accommodate it's decimal structure since 200 is \( 2 \times 2 \times 2 \times 5 \times 5 \) with the highest power seen on \( 2^3 \) is 3 so this will take 3 places to end as a decimal.

The number of places needed for a 10 friendly structure is solely dependent upon the highest power on the 2 OR the 5 in the denominator.

However when we look at \( \frac{5}{6\text{ths}} \) and try to project it into 10ths or 100ths etc it is not friendly to 10, so division of 5 by 6 produces the repetitive pattern block of \(.8333...\) \( \text{6)} \frac{5}{0.000} \)

If you have \( \frac{19}{20\text{ths}} \) then see \( \frac{7}{20\text{ths}} \) as \( \frac{35}{100\text{ths}} \) = .35 then you have 19.35 for 19 and 35:100ths

Now consider returning the decimal to it's rational form. If it's 10 friendly then it can be read back to the rational form. So \(.8 \text{ is 8: tenths or } \frac{8}{10\text{ths}} \text{ or } \frac{4}{5\text{ths}} \).

Similarly \(.85 \text{ can be read as 85: hundredths or } \frac{85}{100\text{ths}} \). But with \(.8333...\) cannot be read back to its’ rational form. So we need an algebraic approach to get it back to \( \frac{5}{6\text{ths}} \).

Let \( N = .83\overline{3} \).

Now we magnify it enough times to get two that match (ie the repetitive structure right at the decimal point.

\( N = .83\overline{3} \)
\( 10N = 8.3\overline{3} \)
\( 100N = 83.3\overline{3} \)

Now take 10N from 100N to see:

\( 100N = 83.3\overline{3} \)
\(- (10N = 8.3\overline{3}) \) to find

\[ 90N = 75 \] which leads to \( \frac{75}{90\text{ths}} \) which is \( \frac{5}{6\text{ths}} \).

The number of magnifications it will take to get the job done is dependent upon two things. The number of displacement issues and the length of the pattern. Displacement means the number of places there are before the pattern begins. You have to magnify it, this number of times to get to the pattern the first time. Then the length of the pattern determines how many times you have to magnify to get to it the second time. So the number of magnifications is a sum of the displacement issues and the length of the pattern.

In the \( .8\overline{3} \) case there's one displacement issue and length of pattern is one digit, so you need 2 magnifications to get the job done.

In the case of \( .5\overline{123} \) there's one displacement and a 3 digit pattern so it will take 4 mags (magnifications) to get this job done.
In the case of \(.512\overline{3}\) it will also take 4 mags for this job since there is 2 digit displacement and 2 digit pattern hence 4 mags.

Each of these belong to the ‘4 mag’ family but for different reasons.

\(\mathbb{N} = \{1, 2, 3, 4, \ldots\}\) \(\mathbb{W} = \{0, 1, 2, 3, 4, \ldots\}\) \(\mathbb{Z} = \{\ldots -2, -1, 0, 1, 2, \ldots\}\) stepping stones

\(\mathbb{Q} = \{\frac{n}{d} / \text{ } n \text{ and } d \text{ are chosen from the integer set provided } d = 0\}\) Pebbles

\(\mathbb{Q}' = \text{irrationals which are predominantly non perfect square roots}. \text{ Sand and mud}\)

\(\mathbb{R} = \text{the union of } \mathbb{Q} \text{ and } \mathbb{Q}' \text{ stepping stones, pebbles & sand and mud. The numberline is dense.}\)

"What does it mean to be irrational?"

Their decimal expansions never end nor ever yield a repetitive pattern block structure. The bulk of the irrationals come from the nonperfectly rootable positions. They lie between any two perfect roots, between any two cube roots etc., though we also have \(\pi\) and 'e' which are also transcendental. Using the following table you can see squares, cubes etc.

<table>
<thead>
<tr>
<th>1 cubed sees</th>
<th>1 squared sees</th>
<th>1 etc</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>8</td>
</tr>
<tr>
<td>3</td>
<td>9</td>
<td>27</td>
</tr>
<tr>
<td>4</td>
<td>16</td>
<td>64</td>
</tr>
<tr>
<td>5</td>
<td>25</td>
<td>125</td>
</tr>
<tr>
<td>6</td>
<td>36</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>49</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>64</td>
<td>It will benefit you to know</td>
</tr>
<tr>
<td>9</td>
<td>81</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>100</td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>121</td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>144</td>
<td></td>
</tr>
</tbody>
</table>

For every square there is some square root which is the return trip ticket. Think of the first column as San Francisco, Ca and the second column (the squares) as Las Vegas, NV and the third column as Denver, Colo. The point is that the roots are nonstop flights back to San Francisco and do not stop in between when flying nonstop to San Francisco.

Next consider what lies between for example the square root of 36 which is 6 and the square root of 49 which is 7. Between these lies the square roots of: 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47 and 48 along with the square roots of 36.1, 36.2, 36.3 etc. etc....

Through these eyes you see why the sand and mud (irrationals) have arrived to surround the stepping stones (integers) and pebbles (rationals) and when all these are at the party the numberline is dense (no gaps which says between any two distinct points there’s an infinite number of points). So “directly next to” makes no sense. I could be standing ‘next’ to you but there is still space between us.

Next let’s approximate a square root without a machine (calculator).

Approximate for example the square root of 91: \(\sqrt{91}\) First see it to lie between:

the square root of 81 = 9 (its lower bound): \(\sqrt{81} = 9\)

So \(\sqrt{91}\) is 9 and some change

the square root of 100 = 10 (its upper bound): \(\sqrt{100} = 10\)
So $\sqrt{91}$ is 9 and some change. Now, between $\sqrt{81}$ and $\sqrt{91}$ is 10 steps VS between $\sqrt{91}$ and $\sqrt{100}$ is 9 steps. Since these distances are very close this says that $\sqrt{91}$ is close to halfway between the two bounds. Therefore a good approximation of $\sqrt{91} \approx 9.5$

Now consider $\sqrt{129}$.

Since the lower bound is $\sqrt{121}$ at 11 and the upper bound is $\sqrt{144}$ at 12

So $\sqrt{129}$ lives between 11 and 12 so it’s 11 and some change. To determine "how much change" look at the distance $\sqrt{129}$ lies from $\sqrt{121}$ is 8 steps VS the distance that $\sqrt{129}$ lies from $\sqrt{144}$ is 15 steps. Since it is closer to $\sqrt{121}$ (it's lower bound) then it's closer to 11 than 12. So it's on the low side, approximately 11.3 So mathematically symbolized we see: $\sqrt{129} \approx 11.3$

To get the next place accuracy then look at $(11.3)^2$ and $(11.4)^2$ to see where is lies between these bounds. This process was tedious for sure.
Second Level Cycle Factoring

Recall that in the simple ONE cycles the order in loading the cycles was immaterial. So given: $x^2 - 10x + 24$ the controls say multiply to 24 and add to 10 so the factors are $(x - 4)(x - 6)$ with directional signs of $(x - 4)(x - 6)$. However you could have listed the factors as $(x - 6)(x - 4)$ without compromising the outcome since the lead coefficient is ONE. When the lead coefficient is ONE the order in loading the cycles does not matter. If the lead coefficient is no longer ONE, then the cycles become more complicated to find since order now matters, which means searching both forward and backward cycles.

Consider $2x^2 + 3x - 5$ which is a very controlled example (the numerical controls are prime). Primes control the available cycles considerably.

Now here, the last sign is negative so the internal parts subtracted to 3x. The same directional controls from the ONE cycles apply but it is not only the end number that creates the proper cycle (provided there is a cycle to do the job). The conversation between 2 & 5 finds the cycles.

So here, insure the $2x^2$ with $(2x \quad)(x \quad)$. Now in the back you have to consider 1 & 5 or 5 & 1. With the 1 & 5 cycle you see: $(2x \ 1)(x - 5)$ and the internals produce 1x and 10x which does NOT subtract to 3x. But with the 5 & 1 cycle you see: $(2x - 5)(x - 1)$, the internals produce 5x and 2x which does satisfy the second job (subtracts to 3x internally). Once again because the first sign is positive the larger size (ie the 5x) takes this direction so the factors are $(2x + 5)(x - 1)$.

Next consider: $6x^2 + 11x - 72$.

To insure the $6x^2$, you have $2x \ & 3x$ OR $x \ & 6x$ whereas the cycles in the back are for 72 are:

1 & 72, 72 & 1
2 & 36, 36 & 2
3 & 24, 24 & 3
4 & 18, 18 & 4
6 & 12, 12 & 6
8 & 9, 9 & 8

Recall, order in loading the cycles now matters which is why for the 72 search, forward and backward cycles must be considered.

So list the cycle possibilities like this:

$$6x^2 + 11x - 72$$

<table>
<thead>
<tr>
<th>front, back</th>
<th>front, back</th>
</tr>
</thead>
<tbody>
<tr>
<td>2x &amp; 3x</td>
<td>1 &amp; 72</td>
</tr>
<tr>
<td>x &amp; 6x</td>
<td>2 &amp; 36</td>
</tr>
<tr>
<td></td>
<td>3 &amp; 24</td>
</tr>
<tr>
<td></td>
<td>4 &amp; 18</td>
</tr>
<tr>
<td></td>
<td>6 &amp; 12</td>
</tr>
<tr>
<td></td>
<td>8 &amp; 9</td>
</tr>
<tr>
<td></td>
<td>72 &amp; 1</td>
</tr>
<tr>
<td></td>
<td>36 &amp; 2</td>
</tr>
<tr>
<td></td>
<td>24 &amp; 3</td>
</tr>
<tr>
<td></td>
<td>18 &amp; 4</td>
</tr>
<tr>
<td></td>
<td>12 &amp; 6</td>
</tr>
<tr>
<td></td>
<td>9 &amp; 8</td>
</tr>
</tbody>
</table>

Now guarantee the $6x^2$ by using $(2x \quad)(3x \quad)$, then you can search the cycles of 72 and this will sufficiently scan all possible cycles, called trial and error which I consider to be stabbing in the dark.

So with the $6x^2$ insured, now search the cycles of 72 for the one that internally subtracts (last sign is negative) to 11x.

This process is eventually successful but also searches through unnecessary cycles.

The way to disqualify cycles is to understand that since there are NO common factors in the original then common factors CANNOT show up in the parts (factors).
This means with \((2x \quad \cdot)(3x \quad \cdot)\) there can be NO 2's in the front factor and NO 3's in the back factor.

This disqualifies most of the possibilities in the 72.

Look at the cycle list for the 72 through the eyes of “with NO 2's in the front and NO 3's in the back “, and the only cycle that survives the disqualification process is 9 & 8.

So putting 9 & 8 in sees \((2x \quad 9)(3x \quad 8)\) which internally produces 27x and 16x which subtracts to 11x (all numerical jobs completed). Lastly for the signs (direction), since the internal mechanism subtracted, the first sign dictates the larger size goes positive, then the final factors are: \((2x + 9)(3x - 8)\) which is \(6x^2 +11x -72\). The internal parts create 27x and -16x which satisfies the secondary job which was internal subtraction to 11x.

The next example shows the power of this disqualification process. Consider: \(12x^2+5x -72\)

The cycles are listed as follows.

<table>
<thead>
<tr>
<th>(12x^2+5x -72)</th>
<th>front,back</th>
<th>front,back</th>
</tr>
</thead>
<tbody>
<tr>
<td>(2x &amp; 6x)</td>
<td>1 &amp; 72</td>
<td>72 &amp; 1</td>
</tr>
<tr>
<td>(3x &amp; 4x)</td>
<td>2 &amp; 36</td>
<td>36 &amp; 2</td>
</tr>
<tr>
<td>(x &amp; 12x)</td>
<td>3 &amp; 24</td>
<td>24 &amp; 3</td>
</tr>
<tr>
<td></td>
<td>4 &amp; 18</td>
<td>18 &amp; 4</td>
</tr>
<tr>
<td></td>
<td>6 &amp; 12</td>
<td>12 &amp; 6</td>
</tr>
<tr>
<td></td>
<td>8 &amp; 9</td>
<td>9 &amp; 8</td>
</tr>
</tbody>
</table>

Now to insure the \(12x^2\), you start with \((2x \quad \cdot)(6x \quad \cdot)\) then since there are NO common factors in the original there can be NO common factors in the parts.

With \((2x \quad \cdot)(6x \quad \cdot)\) locked in, this says NO 2's in the front as well as NO 2's OR 3's in the back. It is more powerful to see 6 in its basic bones(factors) .So searching the cycle list of 72 quickly finds none of these work (because each cycle has a 2 in the front or a 2 in the back) which says \((2x \quad \cdot)(6x \quad \cdot)\) is incorrect.

Next you have \((3x \quad \cdot)(4x \quad \cdot)\) to guarantee the \(12x^2\).

<table>
<thead>
<tr>
<th>(12x^2+5x -72)</th>
<th>front,back</th>
<th>front,back</th>
</tr>
</thead>
<tbody>
<tr>
<td>(3x \quad \text{and} \quad 4x)</td>
<td>1,72</td>
<td>72,1</td>
</tr>
<tr>
<td></td>
<td>2,36</td>
<td>36,2</td>
</tr>
<tr>
<td></td>
<td>3,24</td>
<td>24,3</td>
</tr>
<tr>
<td></td>
<td>4,18</td>
<td>18,4</td>
</tr>
<tr>
<td></td>
<td>6,12</td>
<td>12,6</td>
</tr>
<tr>
<td></td>
<td>8,9</td>
<td>9,8</td>
</tr>
</tbody>
</table>

Now with \((3x \quad \cdot)(4x \quad \cdot)\) there can be NO 3's in the front and NO 2's in the back (4's bones are 2's).

When you search the cycles of 72 through the eyes of NO 3’s in the front and NO 2’s in the back, the only cycle that is not disqualified is the 8 & 9 . So the factors are: \((3x \quad 8)(4x \quad 9)\) which internally produces 32x and 27x which satisfies the internal control that said subtract to 5x. Lastly the directional signs are: \((3x +8)(4x -9)\) since the larger internal control comes from the 32x as opposed to the 27x.
Through these combinatoric eyes you can expeditiously find the correct cycle (if there is one) and not waste time searching through unnecessary cycles.

You can interpret this disqualification process as what qualifies someone for a race. If someone does not get in the race they cannot win for sure. Keep in mind that just because someone gets in a race does not mean they win.
**Simple One Cycle Factoring**

**Simple One Cycles Factoring** must have 3 terms and the lead power MUST be twice the size of the secondary power. So \(x^2 - 10x + 24\) qualifies but \(x^3 - 10x + 24\) does not. However \(x^3 - 10x^2 + 24x\) factors out an ‘x’ to see \(x(x^2 - 10x + 24)\) then factors within.

This type of factoring was once called Cycle factoring for it involves searching for a cycle that satisfies both conditions within. There are two cases and within each case there are two possible directions.

Consider the difference between: \(x^2 - 10x + 24\) VS \(x^2 + 10x - 24\)

\(x^2 - 10x + 24\) is case I, \(x^2 + 10x - 24\) is case II.

---

In case I, the last sign is positive which says the internal parts **ADDED** to the middle number. So \(x^2 - 10x + 24\) actually says I want to multiply to 24 **AND** also add to 10. So when I consider the cycles to do the first job (multiply to 24) they are 1 & 24, 2 & 12, 3 & 8 or 4 & 6. Of these the one that does the second job (add to 10) is the 4 and 6 cycle. So the factors are: \((x-4)(x-6)\). The last thing to identify is the direction (signs).

Since the internal parts added, the signs **MUST** be the SAME and they are whatever the first sign is (in this case both negative). So the factors are \((x-4)(x-6)\).

Similarly in \(x^2 + 10x + 24\) also says: I want to multiply to 24 **AND** add to 10. So the factors are again \((x+4)(x+6)\) but the direction is responding to the first sign which is positive so the factors are \((x+4)(x+6)\). These are the two directions that the first case can take since they both belong to the same cycle family i.e. multiply to 24 **AND** add to 10.

---

In case II, the last sign is negative which says the internal parts **SUBTRACT** to the middle number. So \(x^2 + 10x - 24\) actually says I want to multiply to 24 **AND** also subtract to 10. So when I consider the cycles to do the first job (multiply to 24) they are 1 & 24, 2 & 12, 3 & 8 or 4 & 6. Of these the one that does the second job (subtract to 10) is the 2 and 12 cycle. So the factors are: \((x-2)(x+12)\). The last thing to identify is the direction (signs).

Since the internal parts subtracted, the signs are **DIFFERENT** and the first sign has to follow the larger size (here the 12 rather than the 2.) So the factors are \((x-2)(x+12)\).

Similarly in \(x^2 - 10x - 24\) also says: I want to multiply to 24 **AND** subtract to 10. So the factors are again \((x+2)(x-12)\) but the direction is responding to the first sign which is negative so the factors are \((x+2)(x-12)\). These are the two directions that the second case can take since they both belong to the same cycle family i.e. multiply to 24 **AND** subtract to 10.

---

The last sign is the operational control while the first sign is the directional control. The signs (direction) are the last thing to consider since the structure will fail to factor because there are no cycles that will do BOTH jobs.

Consider \(x^2 + 6x + 7\) which says I want to multiply to 7 and add to 6 which cannot be done since the only cycle for the first job (multiply to 7) are 1 and 7 which cannot satisfy the second job (add to 6). Whereas \(x^2 + 6x - 7\) says I want to multiply to 7 and subtract to 6 so the only
cycle for the first job (multiply to 7) will subtract to 6. Therefore the factors are \((x - 1)(x - 7)\) with signs (direction) going to \((x-1)(x+7)\). Since the first sign is positive it follows the larger number within the factors, the 7 VS the 1.

These cases can later be used to demonstrate reflections of a parabola when the signal (function) of a parabola is set equal to zero to find the x-axis hits. The two directional possibilities within each case are simply copies of the same parabola in different positions.
Simplifying Roots

Recall that the difference between a rational and an irrational is about the ability to round off with accuracy.

First see $\sqrt{81}$ to lie between: the square root of $81 = 9$ (its lower bound): $\sqrt{81} = 9$

So to approximate for example the square root of $91$: $\sqrt{91}$ = 9. and some change

and the square root of $100 = 10$ (its upper bound): $\sqrt{100} = 10$

Now, between $\sqrt{81}$ and $\sqrt{91}$ is 10 steps VS between $\sqrt{91}$ and $\sqrt{100}$ is 9 steps. Since these distances are very close this says that $\sqrt{91}$ is close to halfway between the two bounds. Therefore a good approximation of $\sqrt{91}$ is close to 9.5
the lower bound is $\sqrt{81}$ at 9

Now consider $\sqrt{129}$ is 11. and some change
the upper bound is $\sqrt{144}$ at 12,

To determine "how much change" look at the distance $\sqrt{129}$ lies from $\sqrt{121}$ is 8 steps VS the distance that $\sqrt{129}$ lies from $\sqrt{144}$ is 15 steps. Since it is closer to $\sqrt{121}$ (it's lower bound) then it's closer to 11 than 12. So it's on the low side, approximately 11.3
So mathematically symbolized we see: $\sqrt{129} \approx 11.3$

Though this process will approximate an irrational, technically they go on forever and never reveal a repetitive pattern like rationals do. Rationals like $\frac{1}{2}$ which is .5 or $\frac{5}{6}$ which is .8333... either end or reveal a repetitive pattern which makes it more accurate to round off. With no pattern, the irrationals are less accurate when rounded off. This process was used before mathematical tools were created which can be used to accurately find the square root of anybody.

Recall that you can add or subtract whenever you have the same structures. So you can add 3X & 5X as 8X but you cannot add 3X² & 5X to get 8 of anything. You can add 3 Boxes + 5 Boxes to get 8 Boxes or 3 spheres + 5 spheres to get 8 spheres but can you add 3 Boxes & 5 spheres to get 8 box-spheres? NO!

Similarly you can add $3\sqrt{7} + 5\sqrt{7}$ to get $8\sqrt{7}$ or $5\sqrt{2} - 8\sqrt{2}$ to get $-3\sqrt{2}$ but you cannot add: $3\sqrt{7} + 5\sqrt{2}$ to get 8 of anything. Now sometimes what looks to be different structures is actually the same underneath. So the goal is to locate and access out any and all perfectly rootable parts. $\sqrt{20}$ is $\sqrt{4 \times 5}$ then rooting the 4 finds $2\sqrt{5}$.

Example: $\sqrt{20} + 6\sqrt{45}$ simplifies to $\sqrt{4 \cdot 5} + 6\sqrt{9 \cdot 5}$ now the 4 roots out 2 and the 9 roots out 3 which hits the 6, so you have $2\sqrt{5} + 18\sqrt{5}$ for $20\sqrt{5}$. Now if the structures are not built upon the same bones you cannot make this happen.
Example: $3\sqrt{20} + 7\sqrt{24}$ sees $3\sqrt{4 \cdot 5} + 7\sqrt{4 \cdot 6}$ which leads to $6\sqrt{5} + 14\sqrt{6}$ which cannot be added since it’s boxes and spheres.

Recognize that roots are not splittable over addition NOR subtraction. $\sqrt{25} - 9$ is $\sqrt{16} = 4$ is not $\sqrt{25} - \sqrt{9}$ which is 5-3 = 2

Recall that multiplication is not size sensitive so $3X(4X^2)$ is $12X^3$ whereas you cannot add: $3X + 4X^2$ for 7 of anything.

Example: for multiplication $\sqrt{5} \cdot \sqrt{15}$ leads to $\sqrt{75}$ which is $\sqrt{25 \cdot 3}$ and the 25 roots out as 5 so we have $5\sqrt{3}$

Example: Recall that $5(x+7)$ leads to $5x + 35$. Similarly: $5(\sqrt{2} + 7)$ leads to $5\sqrt{2} + 35$ by distribution.

Example: $(3+\sqrt{5})(3-\sqrt{5})$ leads to $3 \cdot 3 - \sqrt{5}\sqrt{5}$ which is $9 - \sqrt{25} = 9-5 = 4$.

For division: consider $\frac{15}{\sqrt{5}}$. Now to actually divide 15 by $\sqrt{3} \approx 1.7320500757\ldots$ would pose a problem because $\sqrt{3}$ is irrational which means it’s decimal structure never ends nor reveals a repetitive pattern like .223223… by which we could “possibly” round it off so the division can begin. Rounding a decimal has a level of inaccuracy. So to avoid this we rationalize the denominator.

We know that $\sqrt{A} \cdot \sqrt{A}$ leads to $\sqrt{A^2}$ which is A. So using this fact on $\sqrt{3}$ we multiply $\frac{15}{\sqrt{3}}$ by $\frac{\sqrt{3}}{\sqrt{3}}$ so the denominator sees $\sqrt{3}\sqrt{3}$ which is $\sqrt{9} = 3$ and the numerator sees $15\sqrt{3}$ so we now have: $\frac{15\sqrt{3}}{3}$ then reduces to $5\sqrt{3}$ which is more accurately calculated compared to the original division.

Rationalizing the denominator uses the same structure if the denominator is monomial (single term) like $\sqrt{3}$

So for $\sqrt{3}$ use itself $\sqrt{3}\sqrt{3}$ which is $\sqrt{9}$ which goes to 3.

But if the denominator is binomial (two terms) it needs the opposite force.

So for $3 + \sqrt{5}$ you need $3 - \sqrt{5}$ so that it processes as: $(3 + \sqrt{5})(3 - \sqrt{5})$ as $9 - 5$ for 4.

For example: For $\frac{15}{3+\sqrt{5}}$ we multiply by $\frac{3-\sqrt{5}}{3-\sqrt{5}}$ to get $\frac{15(3-\sqrt{5})}{(3+\sqrt{5})(3-\sqrt{5})}$ which leads to $\frac{15(3-\sqrt{5})}{9-5}$ for $\frac{15(3-\sqrt{5})}{4}$

**Fractional powers** are just a compact way of recording the actions of powers and roots in one symbol. So $8^{\frac{2}{3}}$ means $(\sqrt[3]{8})^2$ which processes as $2^2$ or 4. This means that $8^{\frac{2}{3}}$ is 4 in fancy pants.

In fractional exponents the denominator is the root being taken and the numerator is the power on top of that. $8^{\frac{2}{3}}$ could also be interpreted as $\sqrt[3]{8^2}$ which is $\sqrt[3]{64}$ for 4 but it’s best to take the root first then power it since the root of something cuts it down.
The same processes that govern powers in general also apply to fractional powers. Recall $X^2 \cdot X^5$ leads to $X^{15}$ since when multiplying (assuming same base), the powers react by adding. So $X^{\frac{2}{3}}$ also adds the powers but LCD’s are needed.

$X^\frac{2}{3} X^\frac{1}{2}$ becomes $X^\frac{4}{3}$ for $X^\frac{7}{3}$ which is $\sqrt[3]{X^7}$. Changing to common denominators makes the expressions go to the same root so they can have a conversation. $X^\frac{2}{7}$ is $\sqrt[7]{X^2}$ which becomes $X^\frac{4}{7}$ while $X^\frac{1}{7}$ is $\sqrt[7]{X}$ which becomes $X^\frac{3}{7}$ which is $\sqrt[7]{X^3}$. Then, in radical form these are reacting as $\sqrt[3]{X^4} \cdot \sqrt[7]{X^3}$ which is $\sqrt[21]{X^{12}}$ for $\sqrt[3]{X^7}$ which is $X^\frac{7}{3}$.

Under division recall the powers subtract. $\frac{X^{15}}{X^5}$ sees $X^{15-5}$ for $X^{10}$. Similarly $\frac{X^2}{X^\frac{1}{2}}$ sees $X^{\frac{2}{2}-\frac{1}{2}}$ which is $X^\frac{4}{3}$ for $X^\frac{1}{2} = \sqrt{X}$.

When something is being raised to yet another power the powers react by multiplying. Recall: $(X^2)^5$ leads to $X^{10}$.

Similarly $(X^\frac{2}{3})^9$ sees $X^{2\cdot9} = X^6$. In radical form this is: $(\sqrt[3]{X^2})^9$ which is $\sqrt[18]{X^{18}}$ which is $\sqrt[3]{X^3X^3X^3X^3X^3X^3} = X^6$.

It is often easier to process in fractional exponent form rather than radical form.

Equations involving radicals have domain restrictions since square roots cannot be negative in the Reals. These are found by setting the signal found under the radical $>0$ (which means keep it positive). So when you power both sides of this kind of equation it may create extraneous roots (phantoms). Think of a radio signal which sometimes can extend beyond its’ normal range but it is not trustworthy to remain clear consistently. These are phantom signals.

\[ \sqrt{x - 1} \text{ sees } 0 = x - 1 \text{ so } x > 0 \]
\[ 0 = - \]
\[ x \text{-axis hit is at } -1. \]

So setting $= 0$ sees $0 = \sqrt{x + 3}$

Squaring both sides sees $0 = x + 3$ the

which finds the x axis hit of -3
When solving radical equations you must check your solutions in the original before squaring (powering) to detect the phantoms. Notice the difference between:

- \(0 = \sqrt{x - 3}\)
- \(0 = \sqrt{x} + 3\).

In the first one you square \(0 = (\sqrt{x - 3})^2\) and get \(-3 = \sqrt{x}\). In this one you move the 3 first then to get access to the \(x+3\). \(0 = x - 3\) so \(x = 3\). \((-3)^2 = (\sqrt{x})^2\) access to \(x\) since the 3 is immediately accessible. So \(x = 9\).

\[
\sqrt{2x - 1} + 7 = 4 \quad \text{is from} \quad y = \sqrt{2x - 1} + 3 \quad \text{which when set = to 0 sees} \quad 0 = \sqrt{2x - 1} + 3 \quad \text{which leads to:} \quad 0 = \sqrt{2(5) - 1} + 3 \quad \text{says 3+3 = 0 so 5 is extraneous (a phantom).}
\]
The Concept of LCD

The concept of Least Common Denominator (LCD) is best clarified through the eyes of number theory. First it is best to think of the LCD as the smallest contained size needed to add/subtract two fractions.

It can be first developed by what is known as the march of the multiples. Let's say you want the LCD for 18 and 12. By the march of the multiples you can see that:

- 12's multiples are 12, 24, 36, 48, 60, 72 etc
- 18's multiples are 18, 36, 54, 72, 90 etc

So since you want the smallest contained size we want 36 to do the LCD job. However this process will be rather tedious if you want the LCD for 54 and 48. You'd have to construct the multiples to 432 to find the smallest contained size here.

Number theory helps to clarify when the cases are different and why. The LCD responds to the way the structures are related. In number theory there are three different ways that numbers (structures) can react.

1. **RELATIVELY PRIME** says the structures share no common information except the number ONE. This means they have **NOTHING** in common except ONE.

   They are prime relative to each other though they may not be individually prime numbers.

   For example 2 and 5, 8 and 9, 2x and 5y, 3x^2 and 10y. In this case the **LCD** will be their **product**.

   So for 2 and 5 we need 10, for 8 and 9 we need 72, for 2x and 5y we need 10xy, for 3x^2 and 10y we need 30x^3y.

   **Example:**\[
   \frac{2}{5} + \frac{12}{20} = \frac{14}{20} \quad \text{since 5 & 4 are unrelated}
   \]

   \[
   + \frac{3}{4} = \frac{15}{20} \quad \text{the common size is 20}
   \]

   Sum is: \(\frac{27}{20}\)ths for \(\frac{7}{20}\)ths

2. The **LIVE IN** condition says that one structure lives in the other.

   For example: 4 and 12, 6 and 24, 4x and 12x^2, 5xy and 15x^2y

   In this case the **LCD** will be the **larger one** since it serves itself and will also serve any structure contained in it. So for 4 and 12 we need 12, for 6 and 24 we need 24, for 4x and 12x^2 we need 12x^2, for the 5xy and 15x^2y we need 15x^2y. The LCD here is constructed quite differently than the relatively prime case.

   **Example:**\[
   \frac{5}{8} = \frac{15}{24} \quad \text{Since 8 lives in 24, 24 will do the LCD job}
   \]

   \[
   \frac{7}{24} = \frac{7}{24} \quad \text{which simplifies to } \frac{11}{12\text{ths}}
   \]

3. The **OVERLAPPING** case says the structures share information between them but
one does not live in the other.

For example 18 and 24, 54 and 48, 18xy and 24x^2y, 24x^2y and 54xy^2. In this case multiplying them will cause the LCD to be unnecessarily large. This would be like packing 7 suitcases for a weekend trip. Not illegal but certainly not efficient.

In the overlapping case the **LCD has to be designed case by case** since the LCD is dependent upon what the structures share. Using the schematic below expedites the search. Put the numbers in a division box and start dividing out whatever they share.

\[
\begin{array}{c|cc}
 & 18 & 24 \\
6 &   &   \\
\hline
3 & 6 & \\
\end{array}
\]

Here they share a factor of 6. So when the 6 is divided out, this reveals the next level of 3 and 4.

\[
\begin{array}{c|cc}
 & 18 & 24 \\
6 & 3 & 4 \\
\end{array}
\]

The bottom level reveals that the 18 has an extra 3 in it that 24 does not have and 24 has an extra 4 in it that the 18 does not have. Since the LCD is made up of **what they share (on the left)** times **what they don't share (on the bottom)** you get the LCD of 6 times 3 times 4 for 72.

This can also be used to reduce fractions because the **reduced** form of the fraction is located at the **bottom** of the schematic. So if you want to reduce \( \frac{18}{24} \) it is \( \frac{3}{4} \).

So next consider the LCD for 54 and 48. Remember by the march of the multiples approach you'd have to hunt until 432nds for these sizes.

\[
\begin{array}{c|cc}
 & 54 & 48 \\
6 & 9 & 8 \\
\end{array}
\]

So the LCD is 6 times 9 times 8 for 432. What's on the **left** (what they share) times what's on the **bottom** (what they don't share). So if you want to add \( \frac{11}{48} \) and \( \frac{7}{54} \) we will need 432nds to do it.

This schematic can also be used if the task is to reduce fractions. At the bottom of the schematic, you find the reduced form of the fraction. So if you have \( \frac{54}{48} \), it reduces to \( \frac{9}{8} \) or \( 1 \frac{1}{8} \).

The next part of the process involves changing to the new size so the addition (blending) can begin.

\[
\begin{array}{c|cc}
\frac{11}{48} & \frac{7}{54} & \frac{?}{432} \\
\end{array}
\]

Now this process is generally seen through the eyes of division.
For example if you want to change $\frac{4}{5 \text{ths}}$ into 15ths then you ask how many times does 5 go into 15, identifying that it is 3, then multiply the 4 by 3 to see $\frac{12}{15 \text{ths}}$.

\[
\frac{4}{5} = \frac{12}{15}
\]

The problem with this mechanical division process is that it is highly dependent upon the depth and strength of times tables knowledge. I do not know times tables for 48 or 54, do you?

So instead of thinking of it through division eyes think of it as: "What does the new size have that the old one is missing?"  

Now we know, the (bones)factors of 432 are 6, 9 and 8 (off the schematic). So to see in 432nds we need to see that:

- 48 is (6)(8) and 432 is (6)(9)(8) so can you see what's missing to get 48 to become 432? It is 9.

\[
\frac{\frac{11}{48 \text{ths}}}{(6)(8)} = \frac{?}{432 \text{nds}}
\]

When 6 times 8 is accounted for in the (6)(9)(8), this shows the 9 is missing. Then multiply 11 by 9 to see it as $\frac{99}{432 \text{nds}}$.

Similarly looking at:

\[
\frac{7}{54 \text{ths}} = \frac{\frac{7}{432 \text{nds}}}{(6)(9)(8)}
\]

So this time, the 8 is missing which sees $\frac{56}{432 \text{nds}}$ as $\frac{56}{432 \text{nds}}$.

So now we add $\frac{99}{48 \text{ths}}$ and $\frac{56}{48 \text{ths}}$ to get a total of $\frac{155}{432 \text{nds}}$.

By using the bones of the old size compared to the bones of the new size we can identify what is missing and apply that within.

The same insights can then be used algebraically as well. The critical issue is to determine which case applies:

the relatively prime case, the live in case or the overlapping case. The LCD is driven by the personalities of the structures.

5x & 8y are unrelated 8x lives in 24x^2

5x & 8y are unrelated 8x lives in 24x^2

so LCD is 40xy so LCD is 24x^2

\[
\frac{3 \times 8y}{5x} = \frac{24y}{40xy} \quad \frac{5 \times 3x}{8y} = \frac{15x}{24x^2}
\]

\[
\frac{5 \times 5x}{8y} = \frac{25x}{40xy} \quad \frac{7}{24x^2} = \frac{7}{24x^2}
\]

Sum is: $\frac{24y+25x}{40xy}$ Result is: $\frac{15x-7}{24x^2}$

It is critical to understand the concept of reducing fractions algebraically.

If I get an answer of: $\frac{5x^2+10xy}{5x}$ then cancelling the 5x with the 5x^2 without affecting the 10xy is illegal since it causes a severe imbalance.
The concrete example below clarifies.
Consider \( \frac{13}{5} \) as \( \frac{10+3}{5} \)
recognize that \( \frac{13}{5} \) is between 2 & 3 since it is
\( 2 \frac{3}{5} \) or 2 \textbf{and} \( \frac{3}{5} \)ths.
If you cancel the \( \frac{10}{5} \) without affecting the 3 then you see:
2 + 3 which is 5 rather than \( \frac{10}{5} \) and \( \frac{3}{5} \) which is \( 2 \frac{3}{5} \).
If you are thinking about cancelling algebraically then first ask: "If it is connected to the
next term by addition or subtraction then get your cancel hands off it, because you are
about to cause a severe imbalance."
The Concept of Percent

The concept of percent is best approached from a proportion standpoint which will allow you understand who is being compared to whom and for what reason.

Percent means: per hundred or out of 100

in relationship to 100 or related to 100
in comparison to 100 or compared to 100

Any of these interprets what the symbol of % means.

The statement 35% of 80 is 28 is only true because it says 35 compares to 100, the same way that 28 compares to 80. So as a proportion it claims: \( \frac{35}{100} = \frac{28}{80} \)

Now if you reduce the left side by 5, you see 7 out of 20. Similarly if we reduce the right side by 4, you see the same 7 out of 20. The original % statement is true because the comparisons are built upon the same bones. Now to define some useful terms.

The 28 is the section, 80 is the base and the 35% is the rate of comparison.

So % problems are driven by who you seek, namely the section, the base or the rate of comparison.

1. Seeking the section asks: What is 35% of 80? This says: what compares to 80 the same way that 35 compares to 100? The proportion sees: \( \frac{?}{80} = \frac{35}{100} \)

solving this proportion sees \( \frac{(80)(35)}{100} \) which leads to 28.

2. Seeking the base asks: 28 is 35% of what number? This says: 28 compares to something the same way that 35 compares to 100. The proportion sees: \( \frac{28}{?} = \frac{35}{100} \)

solving this proportion sees \( \frac{(28)(100)}{35} \) which leads to 80.

3. Seeking the rate of comparison asks: What percent of 80 is 28? This says: What compares to 100 the same way that 28 compares to 80? The proportion sees: \( \frac{28}{80} = \frac{?}{100} \)

solving this proportion sees \( \frac{(28)(100)}{80} \) which leads to 35.

The critical insight is that these three versions of the question are the same comparison looking for a missing component of the same relationship.

Understanding the comparisons will enable you to interpret word problems through the eyes that see the percent measures how the section compares to the base. %

always measures: how the section compares to the base. \( \frac{\text{section}}{\text{base}} \)

In a section problem, you can use 10% to find 5%, 20%, 30% etc. If you want to find 10% of a number it is the same as dividing by 10. If you know what 10% is, then to find 20% you double 10%. Similarly if you want 5%, you take half of whatever you found 10% to be. So since 10% of 80 is 8, then 20% of 80 is double 8 finding 16. Since 10% of 80 is 8, then 5% of 80 is half of 8 finding 4.

Another interpretation that clarifies who is being compared to whom and for what reason is that percent measures what actually happened against what could have happened. \( \frac{\text{actually happened}}{\text{could have happened}} \)

So in survey problems the base is the number of
people who participate in the survey, then it sectionalizes into those who said yes, those who said no and perhaps those who have no opinion. If you want to measure the % of yes’s then you compare the number who actually said yes to those who could have said yes. So the three problems that come out of a survey situation, could be seeking the section: (those who actually said yes), the base: those who participated in the survey: (could have said yes), or the rate of comparison: (who actually said yes compared to who could have said yes. (% of yes’s)

In a survey of 800 people, 280 said yes to a particular question. What percent said yes to this question?

\[
\frac{\text{actually happened}}{\text{could have happened}} = \frac{280}{800} = \frac{?}{100},
\]

solving this finds 35%.

In a discount problem, two sections are possible: the amount you pay and the amount you save.

So a discount problem application could be looking for the section you save, the base which is the original ticketed amount, or the rate of comparison which is the % you save (i.e. the discount percent). So consider:

You paid $52 for some item. If the original price was $80 then what percent discount did you get?

To measure this you can see: \(\frac{52}{80} = \frac{?}{100}\). Solving this finds you pay 65%, so you save 35% (discount on this item). OR You can recognize that since you pay $52 out of the $80 price, this says that $80-$52 leaves $28 which is the discount (what you save). So measuring \(\frac{28}{80} = \frac{?}{100}\) solves to see 35% directly for the discount (amount saved).

The mathematical machinery is the same in rates of increase or decrease. In fact the machinery does not know whether you gained money or lost money (but you do).

You always measure (compare) the actual change that occurred in comparison to the starting position (which is the number to which the change occurred).

\[
\frac{\text{Actual change}}{\text{Starting position}} = \frac{?}{100}
\]

This measures rate of increase or decrease whichever applies.

So consider: I bought a house in 2001 for $75,000 and sold it in 2006 for $125,000. What % increase is this? (in business this is called appreciation)

Now measure: \(\frac{\text{Actual change}}{\text{Starting position}} = \frac{50,000}{125,000} = \frac{?}{100}\) Solving this finds a 40% increase. Keep in mind that this took 5 years to occur so it is essentially an 8% increase per year.

Now what if: I bought the house in 2006 for 125,000 and had to sell in 2011 for 75,000. What % decrease is this? (in business this is called depreciation).

Now measure: \(\frac{\text{Actual change}}{\text{Starting position}} = \frac{50,000}{75,000} = \frac{?}{100}\)

Solving this finds 66\(\frac{2}{3}\)\% decrease. Once again this occurred over 5 yrs so the average loss is 13\(\frac{1}{3}\)\% or 13.33\% .
Word Problems

First you have to recognize the words that determine which operation is being used. The first operation is the dominant one and any operation after that is a sub operation. Think of it as a set of directions to somewhere. If the directions say: go 5 miles make a right then 2 miles this is not the same as go 2 miles make a right then 5 miles. You can’t interchange the dominant and sub operations without effecting the destination. These are expressions which are not solvable.

For addition:
- This plus that says this + that
- 6 more than a number says: N+6
- The sum (total) of this & that says: This + that
- A number increased by 6 says: N+6

For subtraction:
- This minus that says: This – that
- 6 less than a number says: N-6
- Difference between this & that: This – that
- A number decreased by 6 says: N-6

For multiplication:
- This times that: (this)(that)
- Twice the size of a number: 2N
- Half the size of a number: \( \frac{1}{2}N \) is the same as
- The product of this and that: (this)(that)

For division:
- This divided by that: \( \frac{this}{that} \)
- A number divided by 2: \( \frac{N}{2} \)
- The quotient of this and that:

Example: The sum of twice a number and 5 says : 2N + 5. Notice that the first operation you hear is “the sum” so you have ___ + ____ and when you identify the parts you see 2N + 5

Example: Twice the sum of a number and 5 says 2(N+5) Notice the first operation you hear is twice(something) & the parts are : 2(N+5) is 2N + 10 not the same as 2N+5

Example: The sum of the squares of two numbers says: ____+____ is \( N^2 + M^2 \)

Example: The square of the sum of two numbers says: (something)\(^2\) is \( (N + M)^2 \)

These are again not the same and you can see this by picking two numbers say N= 3 & M= 4 to see: \( N^2 + M^2 \) sees \( 3^2 + 4^2 \) for 9+16= 25 whereas \( (N + M)^2 \) sees \( (3 + 4)^2 \) for \( 7^2 = 49 \)

Word problems are about finding where (if at all) given conditions agree. There are two types of connections (relationships) relating entities. Betweens VS Upons

A between connection tells you how the entities are related to each other.

An upon connection gives some outcome the entities produced but you do not know directly how they are related if they are related at all. Here are some BETWEEN connections: The length of a rectangle is 3 more than twice it’s width says: \( L = 2W + 3 \)
There are six more nickels than dimes. Since nickels has the bigger pile it says: \(N = D + 6\)

One number is 4 times the size of another says: \(O = 4A\)

Here are some **UPON connections**. The perimeter of a rectangle is 66 ft. says: \(2L + 2W = 66\)

The pile of nickels and dimes is worth $3.30 says: \(.05N + .10D = 3.30\) or in pennies: \(5N + 10D = 330\). The sum of two numbers is 25 says: \(O + A = 25\)

**Word problems** have multiple conditions and the task is to find out if they agree and where.

One number is 4 times the size of another says: \(O = 4A\) Their sum is 25 says: \(O + A = 25\)

So we are trying to find where these conditions agree(if they do).

\(O = 4A\) spans out & collects the points that hear it. At the same time \(O + A = 25\) collects points. The list would look like this: \(O=4A\) The list looks like:

\[
\begin{array}{c|c}
A & O \\
\hline
1 & 4 \\
2 & 8 \\
3 & 12 \\
5 & 20 \\
10 & 40 \\
\end{array}
\]

Now look at where they agree. It’s at 5 & 20 so this search and find method would work but would be extremely long and tedious if the number controls are large. So let the algebra do the search & find for you.

We have: \(O = 4A\) and \(O + A = 25\) so take the 4A (which is worth O) and substitute it into the other control which tells them to find and tell you where they agree.

In \(O + A = 25\) replacing \(O\) with \(4A\) sees \(4A + A = 25\) which reveals that \(5A = 25\) so \(A = 5\)

Using either control (they agree here) you can recover the other value. So \(O = 4A\) sees \(O = 4(5)\) for 20. So one number is 5 & the other is 20.

This approach algebraically does not know nor care about what the problem may be describing. Letting the algebra do the search & find becomes language independent.

This same set of controls could have said: The number of orangutans is 4 times the number of alligators.

\(O = 4A\) and the total number of orangutans and alligators is 25. \(O + A = 25\). Then this would solve just like the other set up and find \(A = 5\) (alligators) and \(O = 20\) (orangutans)

There is a basic **difference between solving** a word problem and **setting up** a condition.

**Example:** One number is 4 times another. **Represent** their sum says you cannot solve this
condition since you do not have enough information. The commands: represent, express, express in terms of trigger that you cannot solve this situation. $O = 4A$ so to represent their sum we see: $A + O$ and replace $O$ with $4A$ to see $A + 4A$ which is $5A$.

**Example:** There are six more nickels than there are dimes & if their total worth is $3.30 then how many of each do you have?

The first control says: $N = D + 6$ and the second control says: $5N + 10D = 330$ in pennies.

So replace $N$ in $5N + 10D = 330$ with $D + 6$ to see $5(D + 6) + 10D = 330$ which leads to $15D + 30 = 330$ which finds $15D = 300$ so $D = 20$. Now use this value to recover the # of nickels. $N = 20 + 6 = 26$

**Example:** If there are 6 more nickels than dimes, represent their worth in pennies. Once again you do not have enough information to solve. So using $N = D + 6$ to replace $N$ in $5N + 10D$ with $D + 6$ to see $5(D + 6) + 10D$ which leads to $15D + 30$.

**Example:** The length of a rectangle is 3 more than twice it’s width and the perimeter is 66 ft. Find the dimensions (length & width) of the rectangle.

The first control says: $L = 2W + 3$ and the second control says: $2L + 2W = 66$

So replace $L$ in $2L + 2W = 66$ with $2W + 3$ to see $2(2W + 3) + 2W = 66$ which leads to $6W + 6 = 66$ which finds $6W = 60$ so $W = 10$. Now use this value in either control to find $L$. $L = 2W + 3$ sees $L = 2(10) + 3$ for $23$

**Example:** The length of a rectangle is 3 more then twice it’s width. Represent the perimeter of this rectangle. The first control says: $L = 2W + 3$ and the second control says: $2L + 2W$ so replace $L$ in $2L + 2W = 66$ with $2W + 3$ to see $2(2W + 3) + 2W$ for $6W + 6$.

You can use any letters (variables) you want, but notice that I use letters to represent the situation that remind me of what is being analyzed. I would not use ‘x’ & ‘y’ unless I was dealing with x-rays and yogurt.

Notice that in the between connections one of the variables stands alone. $O = 4A$, $N = D + 6$, $L = 2W + 3$ This makes them ready for the substitution job. The variable that stands alone is the worker bee (the one that does the substitution job) Whereas in the upon connections the variables are clustered and one does not stand alone. These are suited for receiving the substitution.

You can create algebraic connections and the **reverse engineer** the problem.

So if I had: $C = A + 7$ and $3C + 5A = 45$ then $C$ is the substitute as $A + 7$ in $3C + 5A = 45$

Now creating the word problem around the connections might say this:
There are 7 more children than adults in a family at a carnival. The tickets for adults are $5 and tickets for children are $3 and the family spent $45 to go to the carnival. How many adults and children in the family?

When the connections solve to a single result then this one result can do both jobs.

If the connections are not compatible then the problem will dissolve which means you’ll get a false statement.

So consider: \( T = 2S + 1 \) and \( 2T - 4S = 3 \) when I substitute \( 2S+1 \) for \( T \) in \( 2T - 4S = 3 \) this sees: \( 2(2S+1) - 4S = 3 \) which says \( 4S + 2 - 4S = 3 \) which says \( 2 = 3 \) which is false so these conditions cannot be simultaneously met. (incompatible) This means that there is no one who can do both jobs which conflict.

Consider: \( T = 2S + 1 \) and \( 3T - 6S = 3 \) when you substitute \( 2S+1 \) for \( T \) in \( 3T - 6S = 3 \) this sees: \( 3(2S+1) - 6S = 3 \) which says \( 6S + 3 - 6S = 3 \) which says \( 3 = 3 \) which is true so these conditions can be satisfied by anybody. This happens because the one equation is actually a multiple of the other. Multiply \( T = 2S + 1 \) by 3 to see \( 3T = 6S + 3 \) which is \( 3T - 6S = 3 \).

If the connections reveal a true statement like \( 3 = 3 \) then this says anybody can do this job.

Sometimes you have to be ready to move the algebraic furniture around because the conditions are not ready for substitution. The sum of two numbers is 15 and their difference is 1 says: \( O + A = 15 \) and \( O - A = 1 \) so neither condition has one variable standing alone( ready to substitute). So taking \( O - A = 1 \) and shifting the A by addition sees:

\( O - A = 1 \) becomes \( O = A + 1 \) so now you can substitute \( A + 1 \) for \( O \) in \( O + A = 15 \) to see: \( A + 1 + A = 15 \) which leads to \( 2A + 1 = 15 \) so \( 2A = 14 \) which sees \( A = 7 \). Since \( O = A+1 \) with \( A = 7 \) you have \( O \) must be 8.

Example: How much 25% acid must be added to 15 liters of 40% acid to produce a 30% solution? This interprets as: \( A(25\%) + 15(45\%) = 30\%(\text{total}) \). Realize that if you add 1 cup of milk to 2 cups of water you have 3 cups of something. You build the total on the parts involved. So our total is \( A+15 \) with which we hav: \( 25\%A + 45\%(15) = 30\%(A+15) \) Solving this yields: \( 25A + 45(15) = 30(A+15) \) leading to \( A = 50 \)

Example: If I add 10 more litres of an acid solution than glycerine and there is 25% acid and 40% glycerine in the mixture then \( \text{represent} \) the total of the mixture.

\( A = G + 10 \) displays the first control. The second control is \( 25\%A + 40\% G \) so substitute \( G+10 \) for \( A \) in the second control to see \( 25\%(G+10) + 40\% \) to get \( 25\%G + 2.5 + 40\%G \) which leads to \( .25G + 2.5 + .4G \) for \( .65G + 2.5 \). Notice it is not equal to anything
which is why you can solve for the exact amounts.

Later you can see that word problems are verbalizations of geometric paths and you are trying to find out where they intersect (agree) if they do. If they do not intersect it’s because the structures are parallel (have no point of agreement).

If you look closely at: There are 3 less tigers than twice as many snakes: \( T = 2S - 3 \) and there are 4 more tigers than twice as many snakes: \( T = 2S + 4 \). Then substituting \( 2S - 3 \) for \( T \) in the second control it sees: \( 2S - 3 = 2S + 4 \) then solving this sees \( -3 = 4 \) which is false so that says these lines are parallel if graphed on a \( T,S \) axis and have no point of agreement (intersection). This means these two conditions cannot be met simultaneously like asking me to parachute down a mountain and climb down the mountain at the same time. 😊